

Half-BPS cosmic string in $\mathcal{N} = 2$ supergravity in the presence of a dilaton

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ABSTRACT: We construct new half-BPS cosmic string solutions in $D = 4$ $\mathcal{N} = 2$ supergravity compatible with a consistent truncation to $\mathcal{N} = 1$ supergravity where they describe D -term cosmic strings. The constant Fayet-Iliopoulos term in the $\mathcal{N} = 1$ D -term is not put in by hand but is geometrically engineered by a gauging in the mother $\mathcal{N} = 2$ supergravity theory. The coupling of the $\mathcal{N} = 2$ vector multiplets is characterized by a cubic prepotential admitting an axion-dilaton field, a common property of many compactifications of string theory. The axion-dilaton field survives the truncation to $\mathcal{N} = 1$ supergravity. On the string configuration the BPS equations constrain the dilaton to be an arbitrary constant. All the cosmic string solutions with different values of the dilaton have the same energy per unit length but different length scales.

KEYWORDS: Extended Supersymmetry, Supersymmetry Breaking, Supergravity Models.

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1. Introduction

Gauge theory solitons, like magnetic monopoles and cosmic strings, are predicted by Grand Unified Theories (GUT). Their formation in the early universe has nontrivial implications for the subsequent cosmological evolution [1, 2]. While the occurrence of monopoles is strongly limited by observations because they would dominate the energy density of the universe, cosmic strings are still compatible with current data. They are no longer considered as the main seed of structure in the universe in favor of inflation, but they are predicted by GUT and many inflation models.

In this paper we study BPS cosmic string solutions in the context of $D = 4$ $\mathcal{N} = 2$ supergravity [18]. Supersymmetry plays an important role in today's attempts to unify fundamental forces beyond the Standard Model. In a supersymmetric theory, BPS configurations are those that preserve a fraction of the supersymmetries. They constitute interesting probes of the high energy regime of the theory as they are usually protected from quantum corrections. As supergravity theories are supersymmetric local field theories of gravity, it is important to analyze their (BPS) solitonic solutions. The latter could have non-trivial implications for the cosmology of supersymmetry theories. They could also correspond to degrees of freedom of a more fundamental theory.

A particular class of BPS solutions has attracted a lot of attention recently: *D-term strings* [3-18]. They are Nielsen-Olesen string-like solutions of a $D = 4$ Abelian-Higgs model coupled to $\mathcal{N} = 1$ supergravity. The corresponding U(1) gauge symmetry is spontaneously broken by the vacuum expectation value of a Higgs field. The Higgs mechanism is due to the presence of a D -term potential endowed with a constant Fayet-Iliopoulos term (FI term). D -term strings preserve half of the supersymmetries: they are half-BPS objects.¹ Their tension saturate a topological bound, the *Bogomolnyi bound*.

D -term strings were first constructed in $\mathcal{N} = 1$ global supersymmetry [3] and later they were discussed again in [4]. Their coupling to $\mathcal{N} = 1$ supergravity was first investigated in [8] and also by [5–7] in the context of 3 dimensional supergravity. The interest in D -term strings increased after the authors of [9] established that they are half-BPS in $\mathcal{N} = 1$ supergravity and conjectured that they represent the low energy manifestation of fundamental objects in string theory called *D-strings*.² The latter are D-branes with one non-compact spatial dimension [20–24, 19]. Since then several string theory analyses have appeared in the literature that support the conjecture [16, 29–32]. However they are also indications of limitation of the conjecture.³

¹It is also possible to construct half-BPS cosmic strings in $\mathcal{N} = 1$ supergravity which are not of the Nielsen-Olesen type. For example, the *magnetic cosmic strings* of [33] don't require any Higgs fields, but only a gauge field and a D -term endowed with a constant FI term.

²For a review of cosmic strings in superstring theory, see [25–27]. The possibility of having heterotic cosmic strings have been discussed in [28].

³ D -term strings are not expected to reproduce the scattering properties of D-strings. Indeed, gauge theory solitons have a reconnection probability $P \approx 1$, while the same quantity for D-strings has been estimated to take values in the range $10^{-1} \lesssim P \lesssim 1$ [25, 26]. The conjecture was based on the observation that D -term strings were the only BPS saturated strings available in $\mathcal{N} = 1$ supergravity. By now, other BPS strings have been obtained with different stability behaviors. For example semilocal strings [13–16]

As a D -term string requires a constant FI term, we are constrained to Abelian gauge theories since FI terms are only allowed in this case. FI terms are also interesting in cosmology aside from the study of topological defects. In the framework of $\mathcal{N} = 1$ supergravity they can be used to generate a positive cosmological constant, what leads to de Sitter vacua and are crucial for D -term inflation [10].

So far, it has not been possible to identify a mechanism in string/M-theory leading to a constant FI term in $\mathcal{N} = 1$ supergravity after compactification. In order to identify such a mechanism, it can be useful to work in $\mathcal{N} = 2$ supergravity as an intermediate step. Indeed, many compactifications of string theory lead to $\mathcal{N} = 2$ supergravity in 4 space-time dimensions. This is an invitation to identify the $\mathcal{N} = 2$ supergravity actions which lead to $\mathcal{N} = 1$ supergravity with a D -term potential and a constant FI term.

In flat space, the embedding of D -term potential with a constant FI term into $\mathcal{N} = 2$ global supersymmetry is well understood [34]. $\mathcal{N} = 2$ global supersymmetry admits a scalar potential that depends on a triplet of *moment maps* that generalize the D -term of $\mathcal{N} = 1$ susy. In flat space, the triplet of moment maps admits a triplet of FI terms that generalize those of $\mathcal{N} = 1$ susy.

The situation changes drastically when we consider the coupling to gravity. Indeed, $\mathcal{N} = 2$ supersymmetry forbids constant FI terms in the presence of hypermultiplets.⁴ At first sight this seems to exclude a description of $\mathcal{N} = 1$ D -term potential with constant FI terms from $\mathcal{N} = 2$ supergravity in presence of hypermultiplets. However, FI terms in $\mathcal{N} = 1$ supergravity do not need to originate from FI terms in $\mathcal{N} = 2$ supergravity. In particular it is possible that a scalar potential in a $\mathcal{N} = 2$ action, after truncating consistently part of the fields, will correspond to an $\mathcal{N} = 1$ potential with a D -term and a constant FI term.⁵

The first known example of a half-BPS cosmic string in $\mathcal{N} = 2$ supergravity was constructed in [18]. It involves the minimal matter content needed to obtain a half-BPS cosmic string solution in $\mathcal{N} = 2$ supergravity action: one hypermultiplet and one vector multiplet.⁶ The construction of [18] can be seen as a consistent embedding of an $\mathcal{N} = 1$ half-BPS cosmic string into $\mathcal{N} = 2$ supergravity while preserving the half-BPS nature of the object.

It is useful to recall some aspects of $\mathcal{N} = 2$ supergravity in order to understand the construction of [18] and the purpose of the present paper.

The scalar fields in $\mathcal{N} = 2$ supergravity can be seen as coordinates of a *scalar manifold*

$$M = M_V \otimes M_H,$$

where M_V and M_H correspond respectively to the scalar fields of vector multiplets and hypermultiplets. The constraints coming from supersymmetry can then be understood as

and axionic D -term strings [12, 17] have a core radius that can vary in size, a property that is not generally expected for D-strings.

⁴As we shall review later on, this is due to a topological obstruction coming from the geometry of hypermultiplets in $\mathcal{N} = 2$ supergravity [34, 45]

⁵An action is said to be consistently truncated to a reduced action, when any solution of the equations of motion of the reduced action is also a solution of the full action.

⁶The hypermultiplet had to be included to provide the scalar acting as a Higgs field since, for Abelian gauging, supersymmetry forbids the scalars of vector multiplets to be charged under gauge transformations.

geometric conditions on the scalar manifold M . $\mathcal{N} = 2$ supergravity requires M_V to be a Special-Kähler and M_H to be of Quaternionic-Kähler type.

The kinetic terms of the scalar fields are defined by a sigma model with target space the scalar manifold M . In order to respect supersymmetry, all internal symmetries of the theory have to preserve the kinetic terms of the scalar fields and therefore correspond to isometries of the scalar manifold. When a group of these internal symmetries is promoted to be *local group* by letting the transformation parameters of the symmetries to depend on spacetime, it is said that the symmetry group has been *gauged*. A gauging introduces new couplings between scalar and gauge fields coming from the usual *minimal coupling* which consists of replacement of derivatives by covariant derivatives. When a supergravity theory is gauged, more couplings have to be included in the action to preserve supersymmetry. In particular, the supersymmetry transformations of the fermions are modified by the so-called *fermionic shifts* that also play the role of mass matrices for the fermions. In $\mathcal{N} = 2$ supergravity, a gauging also requires the introduction of a scalar potential quadratic in the fermionic shifts.

In $\mathcal{N} = 1$ supergravity, the D -term corresponds to the part of the scalar potential generated by a gauging whereas the F -term is due to a superpotential. The D -terms are fermionic shifts for the gaugini. The superpotential gives masses to the gravitino and the chiral fermions. In $\mathcal{N} = 2$ supergravity (like in all extended supergravity theories), gauging is the only way to generate a scalar potential.⁷

In the present paper we shall only be concerned with Abelian gauging. When considering Abelian gauging, only the isometries of the quaternionic-Kähler manifold M_H are relevant as the scalar fields of the special manifold M_V , being part of vector multiplets, are neutral under Abelian symmetries. A gauging of isometries of the quaternionic-Kähler manifold M_H contributes to the fermionic shifts through the Killing vectors representing the isometries and a triplet of so-called *moment-maps*.

In [18], after choosing the scalar manifold M , a specific compact $U(1)$ symmetry of the quaternionic manifold was gauged to generate a scalar potential. The ansatz for the cosmic string solution used only a subset of the fields in such a way that the solution was also valid for an $\mathcal{N} = 1$ supergravity model coupled to one vector fields and one chiral multiplet. In other words, the cosmic string ansatz is compatible with a consistent truncation of the theory to $\mathcal{N} = 1$ supergravity. However, supersymmetry is not spontaneously broken to $\mathcal{N} = 1$ supergravity and the full $\mathcal{N} = 2$ supersymmetry is preserved in the vacuum at spatial infinity. From a $\mathcal{N} = 2$ point of view, the use of an ansatz related to a consistent truncation to $\mathcal{N} = 1$ supergravity in [18] is not required but is useful to simplify the calculation, as $\mathcal{N} = 2$ BPS equations are in general much more difficult to solve than those of $\mathcal{N} = 1$ theories. Alternatively, one can consider that the construction of [18] is a consistent embedding of a cosmic string solutions of an $\mathcal{N} = 1$ supergravity model into an $\mathcal{N} = 2$ supergravity model while preserving the BPS property of the solution. Consistent truncations of $\mathcal{N} = 2$ supergravity are studied systematically in [46–48].

⁷When hypermultiplets are not present, it is possible to include a triplet of $\mathcal{N} = 2$ constant Fayet-Iliopoulos term without any gauging.

The bosonic part of the reduced $\mathcal{N} = 1$ action of [18] is

$$e^{-1}\mathcal{L} = \frac{1}{2}R - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2(\text{Im}\Phi)^2}\nabla_\mu\Phi\nabla^\mu\bar{\Phi} - 2g^2\left[\frac{|\Phi|^2+1}{\text{Im}\Phi} - \xi\right]^2,$$

where $F_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu$, $\nabla_\mu\Phi = \partial_\mu\Phi - 2gW_\mu(\Phi^2+1)$. There is no superpotential, the kinetic term of the vector fields has a trivial metric and the D -term is $D = \frac{|\Phi|^2+1}{\text{Im}\Phi} - \xi$, where ξ is the FI term.

The main purpose of the present paper is to enlarge the type of $\mathcal{N} = 2$ supergravity theories that can generate constant FI term in $\mathcal{N} = 1$ supergravity. The generalization to other quaternionic manifolds that are symmetric spaces is straightforward. However, the special geometry was a very particular case. In $\mathcal{N} = 2$ supergravity, special geometry determines the couplings of the vector fields of the theory. The special geometry used in [18] corresponds to the so-called *minimal special geometry*.

In special geometry, the scalar manifold is a Kähler-Hodge manifold.⁸ However, the Kähler potential \mathcal{K} is not the fundamental object. It is computed in terms of a so-called *symplectic section* (Z^I, F_I) as

$$\mathcal{K} = -\log[-i(Z^I\bar{F}_I - F_I\bar{Z}^I)].$$

where $I = 0, \dots, n_V$ and n_V is the number of vector multiplets. The section depends on the scalar fields of the vector multiplets.

The symplectic section is subject to symplectic transformations. The latter are not symmetries of the Lagrangian but of the equations of motion and Bianchi identities of the vector fields. The Kähler potential is also a symplectic invariant and therefore the geometry of the special manifold is invariant under symplectic rotations. However, the scalar potential and the metric of the kinetic term of the vector fields are not invariant under symplectic rotations.

A symplectic section is said to admit a *prepotential* when it is possible to define an holomorphic function $F(Z)$ depending only on the first half of the symplectic section such that

$$F_I = \frac{\partial}{\partial Z^I}F.$$

$F(Z^I)$ is called the *prepotential* and is required to be homogeneous of order two (that is $F(\lambda Z) = \lambda^2 F(Z)$, for any complex λ). It is always possible to rotate a given symplectic section into one that admits a prepotential [35]. It follows that prepotentials provide a simple way to classify families of special geometries. Such a classification is enough to discuss aspects of special geometry that are invariant under symplectic rotations like for example the geometry of the special manifold. However, more generally, the prepotential is not enough to distinguish two physically different supergravity theories once the theory is gauged. For example, spontaneous supersymmetry breaking to $\mathcal{N} = 1$ supersymmetry

⁸A Kähler-Hodge manifold is a Kähler manifold with a Kähler form defining an integer cohomology. In $\mathcal{N} = 1$ supergravity, the scalar manifold can be any Kähler-Hodge manifold. For a special Kähler-manifold, the Kähler two-form has an even integer cohomology.

is only possible when the symplectic section does not admit a prepotential [36]. Moreover, as the scalar potential is not a symplectic invariant in $\mathcal{N} = 2$ supergravity, the stability of vacua can be modified by a symplectic rotation as illustrated in the study of de Sitter vacua in [37].

Minimal special geometry corresponds to a quadratic prepotential F :

$$F(Z) = -\frac{i}{2}(Z^0 Z^0 - Z^a Z^a), \quad a = 1, \dots, n_V.$$

This is the so-called *minimal prepotential* and the corresponding special manifold is $\frac{SU(1, n_V)}{U(n_V)}$.

Minimal special geometry has many virtues which makes it an interesting candidate to construct cosmic string solutions. In particular, all the scalar fields of the special manifold can be consistently truncated on the string configuration, in such a way that the string solution is based on the minimal amount of matter field required: one vector field and one scalar field. The metric of the kinetic terms of the vector fields is then trivial.

To enlarge the family of supergravity models related to $\mathcal{N} = 1$ supergravity with a D -term, and admitting a constant FI term, we shall study the mechanism described in [18] with a special geometry that is related to a *cubic prepotential*:

$$F = id_{IJK} \frac{Z^I Z^J Z^K}{Z^0}, \quad I, J, K = 1, \dots, n_V,$$

where d_{IJK} are real constant coefficients symmetric in their three indices.

Realizing the construction of [18] with a special geometry related to a cubic prepotential is a natural direction for generalization in view of all the interesting models that require this type of special geometry. Cubic prepotentials are classified in [52]. They characterize the $\mathcal{N} = 2$, $D = 4$ supergravity theories coming from $\mathcal{N} = 2$, $D = 5$ supergravity theories. They are also the special geometry of many compactifications of string theory like for example type *II* string theories compactified on Calabi-Yau three-folds [45, 49, 35] and the Heterotic string compactification on $K3 \times T^2$ [45, 49, 50] and the $D3 - D7$ model (type *IIB* on $K3 \times T^2 / \mathbb{Z}_2$ in presence of open string moduli) [38 - 40]. As we shall see, the change of the symplectic section will imply important differences in the qualitative behaviour of the resulting cosmic string solution. Our choice of special geometry will contain an axion-dilaton field $S = a - ie^\rho$ parametrizing the Kähler manifold $\frac{SU(1,1)}{U(1)} = \frac{SO(2,1)}{SO(2)} = \frac{SL(2, \mathbb{R})}{SO(2)}$ which corresponds to the complex half-plane.

We will consider the special geometry based on the coset space

$$M_V = ST[2, n] \equiv \frac{SU(1, 1)}{U(1)} \times \frac{SO(2, n)}{SO(2) \times SO(n)},$$

with the so-called *Calabi-Vesentini symplectic section* [45, 49] well-known from different compactifications of string theory [38, 39].

On the quaternionic side, the analysis can be done with any normal quaternionic manifold. To avoid complications that are not essential to the construction, we shall consider the quaternionic manifold of (quaternionic) dimension one $M_H = \frac{SO(4,1)}{SO(4)}$. The

model is then simple enough to be analyzed in detail and our results can be compared with those of [18].

We perform the same compact Abelian gauging as in [18] and use a consistent truncation ansatz to obtain cosmic string solutions. The reduced theory is an $\mathcal{N} = 1$ supergravity coupled to a vector multiplet and two chiral multiplets corresponding to the axion-dilaton and Higgs fields.

We shall see that with the choice of the Calabi-Vesentini symplectic section, the $\mathcal{N} = 2$ scalar potential is bounded from below as long as we gauge a vector of negative signature with respect to the metric η_{IJ} used to define the Calabi-Vesentini basis. In the case of minimal special geometry it was possible to truncate all the scalar fields of the vector multiplets on the string configuration.

The bosonic part of the truncated $\mathcal{N} = 1$ action, as defined above, is:

$$e^{-1}\mathcal{L} = \frac{1}{2}R + \frac{1}{4}(\text{Im } S)F^{\mu\nu}F_{\mu\nu} + \frac{e^{-1}}{8}(\text{Re } S)\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} - \frac{1}{4(\text{Im } S)^2}\partial_\mu S\partial^\mu \bar{S} - \frac{1}{2(\text{Im } \Phi)^2}\nabla_\mu\Phi\nabla^\mu\bar{\Phi} + \frac{2g^2}{\text{Im } S}\left[\frac{|\Phi|^2+1}{\text{Im } \Phi} - \xi\right]^2,$$

where $F_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu$, $\nabla_\mu\Phi = \partial_\mu\Phi - 2gW_\mu(\Phi^2 + 1)$. The holomorphic function that defines the kinetic term of the vector field is $f = iS$ where $S = a - ie^\rho$ is the axion-dilaton field. The Kähler-Hodge manifold is

$$\left(\frac{\text{SO}(2,2)}{\text{SO}(2) \times \text{SO}(2)}\right)_{S,\Phi} = \left(\frac{\text{SU}(1,1)}{\text{U}(1)}\right)_S \times \left(\frac{\text{SU}(1,1)}{\text{U}(1)}\right)_\Phi,$$

Each of the two complex scalar fields (S and Φ) parametrize a $\frac{\text{SU}(1,1)}{\text{U}(1)}$ factor. The Kähler-potential is $\mathcal{K} = -\log [i(S - \bar{S})] - 2\log [-i(\Phi - \bar{\Phi})]$. There is no superpotential and the D -term is $D = \frac{|\Phi|^2+1}{\text{Im } \Phi} - \xi$, where ξ is the FI term. The gauge symmetry is $\delta\Phi = 2g(\Phi^2 + 1)$.

We shall show that it is possible to obtain half-BPS cosmic string solutions which solve the full $\mathcal{N} = 2$ equations of motion. These string solutions are defined in the absolute minimum of the potential, which is of Minkowski type and preserves the full $\mathcal{N} = 2$ supersymmetry.

The $\mathcal{N} = 2$ BPS equations imply that the axion-dilaton is an arbitrary constant. For a fixed winding number, we shall show that all the half-BPS solutions with different values of the dilaton have the same energy per unit length.

Solutions with different value of the dilaton can be distinguishable by their characteristic length scales. Indeed, the masses of the fields that define the string solution, m_W for the vector and m_Φ for the scalar, depend explicitly on the dilaton field:

$$\frac{1}{m_W^2} \equiv l_W^2 \propto -\text{Im } S = e^\rho, \quad \frac{1}{m_\Phi^2} \equiv l_\Phi^2 \propto -\text{Im } S = e^\rho.$$

As a consequence we obtain a one parameter family of string solutions, degenerate in energy but with varying core radius.

The structure of the paper is the following. In section 2 we review the mathematical tools needed for $\mathcal{N} = 2$ supergravity: special and quaternionic geometry, the gauging of

isometries and the $\mathcal{N} = 2$ supersymmetry transformations. At the end of the same section we also discuss consistent reduction of supersymmetry, and we explain how the constant FI term is obtained in the reduced $\mathcal{N} = 1$ supergravity model. In section 3 our model is presented. We give the choice of special and quaternionic geometry together with the gauging, which determines the scalar potential. In section 4 we study the vacua of the potential and the $\mathcal{N} = 1$ reduced theory is presented. Next, we construct the string ansatz and we solve the BPS equations. At the end of this section some of the properties of the string are discussed. Finally we review our results in section 5.

2. Review of $\mathcal{N} = 2$ supergravity

2.1 Overview

In $D = 4$ dimensional spacetime with Minkowski signature, the supersymmetry generators are Majorana spinors. A Majorana spinor admits four degrees of freedom and can be decomposed into two chiral spinors of opposite chirality. In four dimensional spacetime, a \mathcal{N} extended supersymmetric theory admits $4\mathcal{N}$ supersymmetric generators organized into \mathcal{N} Majorana spinors. $\mathcal{N} = 2$ supersymmetry requires two Majorana spinors that represent a total of 8 independent supersymmetry generators. We shall work with the corresponding four chiral spinors that we denote (ϵ^i, ϵ_i) . The index $i = 1, 2$ labels the original Majorana spinors and the position of that index represents the chirality. ϵ^i is a left-handed spinors while ϵ_i is right-handed:

$$\epsilon^i = \frac{1}{2}(1 + \gamma_5)\epsilon^i, \quad \epsilon_i = \frac{1}{2}(1 - \gamma_5)\epsilon_i. \tag{2.1}$$

We follow the notation and conventions of [41, 42]. Charge conjugation relates the two chiral projections of a given Majorana spinor. We shall use the same convention for other chiral spinors.

2.1.1 Supersymmetry and geometry

The scalar fields present in supersymmetric multiplets can be seen locally as coordinates of a manifold, (the *scalar manifold* M), whose geometry is restricted by supersymmetry. The latter splits into a direct product of scalar manifolds corresponding to different types of multiplets present in the theory.

When there is an action, the kinetic terms of the scalar fields ϕ^r define a sigma model with target space the scalar manifold M

$$\mathcal{L}_{\phi, \text{kinetic}} = -\frac{1}{2}g_{rs}(\phi)\partial_\mu\phi^r\partial^\mu\phi^s. \tag{2.2}$$

As we have said before, the scalar fields ϕ^r can be seen as local coordinates of the scalar manifold M . From this point of view, $g_{rs}(\phi)$ is interpreted as a metric defined on M . Thus the scalar manifold M has the structure of a Riemannian space.

The supersymmetry transformations will involve a vielbein on the scalar manifold M . The reality conditions due to the type of spinors that are used, the R-symmetry and

Gravity multiplet	vielbein	e_μ^a	$i = 1, 2$ $\mu, a = 0, \dots, 3$
	gravitini	$\psi_\mu^i, \psi_{i\mu}$	
	graviphoton	“ W_μ^0 ”	
Vector multiplet	gauge fields	W_μ^α	$\alpha = 1, \dots, n_V$
	gaugini	$\lambda_i^\alpha, \lambda_\alpha^i$	
	scalars	z^α	
Hypermultiplets	hyperscalars	q^X	$X = 1, \dots, 4n_H$ $A = 1, \dots, 2n_H$
	hyperini	ζ^A, ζ_A	

Table 1: Field content of $\mathcal{N} = 2$ supergravity coupled to n_V vector multiplets and n_H hypermultiplets. The physical graviphoton is not necessarily W_μ^0 but whatever field appears in the supersymmetric transformation of the gravitini through its field strength $T_{\mu\nu}$. The latter is a linear combination of field strengths of all the gauge fields W_μ^I present in the theory ($I = 0, \dots, n_V$) with coefficients that depend on the scalar fields z^α of vector multiplets. The couplings of all the gauge fields W_μ^I and the scalar fields z^α is controlled by *special geometry*.

the closure of the supersymmetry algebra will impose restrictions on the vielbein. These restrictions will be understood as geometric conditions on the scalar manifold.

In $\mathcal{N} = 2$ supergravity coupled to n_V vector multiplets and n_H hypermultiplets, the scalar manifold is a direct product

$$M = M_V \otimes M_H, \tag{2.3}$$

where M_V and M_H are respectively the scalar manifold of vector and hypermultiplets. The restrictions coming from supersymmetry impose that M_V is a so-called *special manifold*, whereas M_H is a *quaternionic manifold*.

The field content of a $\mathcal{N} = 2$ supergravity theory coupled to n_H hypermultiplets and n_V vector multiplets is displayed in table 2.1, where we also fix our notation.

For convenience, the $\mathcal{N} = 2$ supersymmetry transformations will be reviewed later on, after the geometry of the scalar manifold and the gauging of isometries have been discussed.

2.2 Vector multiplets and special geometry

We consider $\mathcal{N} = 2$ supergravity coupled to n_V vector multiplets [60, 61, 50]. For a modern review, see [45, 35, 51].

Since the gravity multiplet contains a vector field, the *graviphoton*, this theory admits $n_V + 1$ vector fields W_μ^I where $I = 0, \dots, n_V$. The n_V vector multiplets contain as well n_V complex scalar fields z^α , $\alpha = 1, \dots, n_V$ parametrizing a Kähler manifold.

The kinetic terms of the scalar and vector fields are:

$$\begin{aligned}
 e^{-1}\mathcal{L}_{\text{vector}} &= \frac{1}{4}\text{Im}(\mathcal{N})_{IJ}F^{I\mu\nu}F_{\mu\nu}^J - \frac{1}{8}e^{-1}\text{Re}(\mathcal{N})_{IJ}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}^IF_{\rho\sigma}^J - g_{\alpha\bar{\beta}}\partial_\mu z^\alpha\partial^\mu\bar{z}^\beta, \\
 &= \frac{1}{2}\text{Im}(\mathcal{N}_{IJ}F_{\mu\nu}^{+I}F^{+\mu\nu J}) - g_{\alpha\bar{\beta}}\partial_\mu z^\alpha\partial^\mu\bar{z}^\beta,
 \end{aligned}
 \tag{2.4}$$

where $F_{\mu\nu}^{\pm I}$ is the self-dual combination:⁹

$$F_{\mu\nu}^{\pm I} = \frac{1}{2}\left(F_{\mu\nu}^I \mp \frac{1}{2}ie\epsilon_{\mu\nu\rho\sigma}F^{I\rho\sigma}\right), \quad F_{\mu\nu}^I = \partial_\mu W_\nu^I - \partial_\nu W_\mu^I.
 \tag{2.5}$$

The matrix \mathcal{N}_{IJ} is a function of the scalar fields z^α . Its real and imaginary parts generalize the inverse of the coupling constant and the θ -parameter familiar from the Abelian gauge theory:

$$e^{-1}\mathcal{L} = -\frac{1}{4g^2}F^{\mu\nu}F_{\mu\nu} + e^{-1}\theta\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}.
 \tag{2.6}$$

The couplings of vector multiplets to $\mathcal{N} = 2$ supergravity are elegantly expressed by *special geometry*. The latter relies heavily on the existence of *duality transformations* for vector fields in supersymmetric theories. Duality transformations generalize the electric-magnetic duality of Maxwell's equations without sources.

The coupling matrix \mathcal{N}_{IJ} undergoes the following fractional transformation:

$$\mathcal{N} \implies (C + D\mathcal{N})(A + B\mathcal{N})^{-1},
 \tag{2.7}$$

under a duality transformation given by a general invertible linear operator $\mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2n_V + 2, \mathbb{R})$ where A, B, C, D are $(n_V + 1) \times (n_V + 1)$ real matrices.

The action of the duality transformations on the scalar fields is much more transparent once we introduce a *symplectic section* which depends on the scalar fields of vector multiplets and transforms linearly under symplectic rotations. Special geometry can be completely defined in terms of this symplectic section.

The symplectic section is given by:

$$v = \begin{pmatrix} Z^I \\ F_I \end{pmatrix}, \quad I = 0, \dots, n_V
 \tag{2.8}$$

and is endowed with a symplectic scalar product

$$\langle v|\bar{v} \rangle = -v^T \begin{pmatrix} \mathbf{0}_n & -\mathbf{1}_n \\ \mathbf{1}_n & \mathbf{0}_n \end{pmatrix} \bar{v}.
 \tag{2.9}$$

Here Z^I and F_I are functions of the coordinates z^α ($\alpha = 1, \dots, n_V$) of the scalar fields of the vector multiplets. Recall that the index I runs from 0 to n_V where n_V is the number of vector multiplets whereas $\alpha = 1, \dots, n_V$ because the graviphoton that appears in the

⁹In our convention the Levi-Civita tensor satisfies $\epsilon_{0123} = 1$.

graviton multiplet is not related to any scalar fields. This is compensated by the freedom to re-scale the symplectic section: the symplectic section is a projective section.

Symplectic rotations act linearly on the symplectic section as:

$$\begin{pmatrix} Z^I \\ F_I \end{pmatrix} \implies \mathcal{S} \begin{pmatrix} Z^I \\ F_I \end{pmatrix}. \tag{2.10}$$

A special manifold is a Kähler manifold in which the Kähler potential is not a fundamental quantity but is given by the following symplectic invariant expression:

$$\mathcal{K} = -\log(-i\langle v|\bar{v}\rangle) = -\log[-i(Z^I \bar{F}_I - F_I \bar{Z}^I)]. \tag{2.11}$$

Here we see that the freedom to re-scale the symplectic section corresponds to a Kähler transformations.

In special geometry, the kinetic terms of both scalar and vector fields are computed from the symplectic section as follow

$$g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} \mathcal{K}(z, \bar{z}) = i\langle \mathcal{D}_\alpha v | \mathcal{D}_{\bar{\beta}} \bar{v} \rangle, \quad \mathcal{N}_{IJ} \equiv \left(F_I \bar{\mathcal{D}}_{\bar{\alpha}} \bar{F}_I \right) \left(Z^J \bar{\mathcal{D}}_{\bar{\alpha}} \bar{Z}^J \right)^{-1}. \tag{2.12}$$

Here the covariant derivatives are defined by

$$\mathcal{D}_\alpha v = \partial_\alpha v + (\partial_\alpha \mathcal{K})v, \quad \mathcal{D}_{\bar{\alpha}} \bar{v} = \partial_{\bar{\alpha}} \bar{v} + (\partial_{\bar{\alpha}} \mathcal{K})\bar{v}. \tag{2.13}$$

When the theory is gauged, the electric-magnetic duality is explicitly broken by the introduction of electric charges. In particular, the scalar potential generated by the gauging is not symplectic invariant.

2.2.1 Special geometry and prepotentials

A special geometry is said *to admit a prepotential* when the lower component of the symplectic section (the variable F_I) can be expressed as derivative of a scalar function $F(Z)$ depending only on the upper part of the symplectic section (Z^I):

$$F_I = \frac{\partial}{\partial Z^I} F(Z). \tag{2.14}$$

$F(Z)$ is restricted to be an homogeneous function of second degree in the Z^I fields and is called the *prepotential*.

Although prepotentials are not necessary to define special geometry, they provide a handy way to classify special manifolds as any symplectic section can be rotated to a section admitting a prepotential [35]. If one is interested only in symplectic invariant quantities, working only with prepotential is not a restriction. This is a practical approach to the classification of special geometry when we consider only the Riemannian geometry defined by the scalar fields as the Kähler potential, the metric and the Riemann tensor are all symplectic invariant.

2.2.2 Minimal and very special geometry

Minimal special geometries correspond to quadratic prepotential defined with a metric η_{IJ} of signature $(1, n)$:

$$F(Z) = -iZ^I \eta_{IJ} Z^J, \quad \eta_{IJ} = \begin{pmatrix} 1 & 0_{1 \times n} \\ 0_{n \times 1} & -\mathbf{1}_n \end{pmatrix}, \quad (2.15)$$

called the *minimal prepotential*. The corresponding special manifold is $\frac{\text{SU}(1,n)}{\text{U}(1)\text{SU}(n)}$.

Very special Kähler geometries are characterized by cubic prepotentials

$$F(Z) = id_{IJK} \frac{Z^I Z^J Z^K}{Z^0}, \quad (2.16)$$

where d_{IJK} is a real symmetric tensor.

Very special Kähler geometries can be obtained by dimensional reduction from 5 dimensional supergravity theories and are known to admit flat potentials. They are also familiar in string theory where they occur in many different compactifications as for example in toroidal compactifications of the heterotic string with possible Wilson lines [45], in compactifications of type *II* string theories on Calabi-Yau threefolds, in compactification of type *II* string theories on orientifolds like $K_3 \times T^2/\mathbb{Z}_2$ in the presence of *D3* and *D7* branes [38, 39].

Although unusual, minimal special geometry is not incompatible with string theory. To the best of our knowledge, there is so far only one case in which it occurs in string theory [54]. This are the $\mathcal{N} = 2$ vacua coming from the $\mathcal{N} = 3$ flux compactification on T^6/\mathbb{Z}_2 studied by Frey and Polchinski [55].

In $\mathcal{N} = 2$ compactifications of string theory, it is important for phenomenological reasons to be able to perform a partial spontaneous supersymmetry breaking to $\mathcal{N} = 1$. As partial supersymmetry breaking is only possible with symplectic sections that do not admit a prepotential [36], minimal and very special geometries usually occur in these types of sections that are related to a prepotential only after a symplectic rotation.

A classification of special manifolds was presented by Cremmer and van Proeyen in [53].

We shall provide some examples to illustrate some subtleties of special geometry. The first example illustrates that the same manifold can be endowed with different types of special geometries. This is the content of table 2, where we review the case of the symmetric space $\frac{\text{SU}(1,1)}{\text{U}(1)}$ which is a Kähler manifold of complex dimension one.

The second example [51] present a symplectic section that is not derived from a prepotential. It is obtained by a symplectic rotation from a minimal special geometry. It will also illustrate how a symplectic section can modify the kinetic terms of the vector fields.

Let us consider the minimal prepotential $F = -iZ^0 Z^1$.¹⁰ The corresponding symplec-

¹⁰This is related to the usual prepotential of minimal special geometry $F = i(Z^0 Z^0 - Z^1 Z^1) = -i(Z^0 + Z^1)(Z^0 - Z^1)$ by the redefinitions $Z^0 \implies Z^0 + Z^1$ and $Z^1 \implies Z^0 - Z^1$.

Prepotential	Symmetric space
$F(X) = -i[(X^0)^2 - (X^1)^2]$	$\frac{SU(1,1)}{U(1)}$
$F(X) = i\frac{(X^1)^3}{X^0}$	$\frac{SU(1,1)}{U(1)}$
$F(X) = -4\sqrt{X^0(X^1)^3}$	$\frac{SU(1,1)}{U(1)}$

Table 2: In this table, all three special manifolds correspond to the same coset space $\frac{SU(1,1)}{U(1)}$. The first one admits a quadratic prepotential and therefore corresponds to the minimal special geometry. The second one is a very special manifold as it admits a cubic prepotential. The last one is related to the second one by a symplectic rotation [51, 53].

tic section is

$$v = \begin{pmatrix} Z^0 \\ Z^1 \\ -iZ^0 \\ -iZ^1 \end{pmatrix} = \begin{pmatrix} 1 \\ z \\ -iz \\ -i \end{pmatrix}, \quad z = \frac{Z^1}{Z^0}. \tag{2.17}$$

The Kähler potential is $\mathcal{K} = -\log 2(z + \bar{z})$. The kinetic matrix for the vector is

$$\mathcal{N} = \begin{pmatrix} -iz & 0 \\ 0 & -\frac{i}{z} \end{pmatrix}. \tag{2.18}$$

The bosonic part of the $\mathcal{N} = 2$ supergravity coupled to a vector multiplet with a special geometry defined by the previous symplectic section is

$$e^{-1}\mathcal{L}_{\text{bosonic}} = \frac{1}{2}\text{R} + \frac{\partial_\mu z \partial^\mu \bar{z}}{(z + \bar{z})^2} - \frac{1}{2}\text{Re} [z(F_{\mu\nu}^{+0})^2 + z^{-1}(F_{\mu\nu}^{+1})^2]. \tag{2.19}$$

After a symplectic rotation with $\mathcal{S} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ we have

$$\tilde{v} = \mathcal{S}v = \begin{pmatrix} 1 \\ i \\ -iz \\ z \end{pmatrix}, \quad \tilde{\mathcal{N}} = \begin{pmatrix} iz & 0 \\ 0 & -iz \end{pmatrix}. \tag{2.20}$$

The new symplectic section cannot be derived from a prepotential as the upper part of the section does not even have a dependence on the scalar field z . The bosonic part of the action after the symplectic rotation is

$$e^{-1}\tilde{\mathcal{L}}_{\text{bosonic}} = \frac{1}{2}\text{R} + \frac{\partial_\mu z \partial^\mu \bar{z}}{(z + \bar{z})^2} - \frac{1}{2}\text{Re} [z(F_{\mu\nu}^{+0})^2 + z(F_{\mu\nu}^{+1})^2]. \tag{2.21}$$

2.3 Hypermultiplets and quaternionic-Kähler geometry

In four dimensional spacetime with the usual Minkowski signature, an hypermultiplet is composed of four real scalar fields and two Majorana spinors. As a Majorana spinor can be decomposed into two chiral spinors of opposite chirality, one can describe n_H hypermultiplets in terms of $4n_H$ real scalar fields q^X ($X = 1, \dots, 4n_H$) and $2n_H$ chiral spinors ζ^A ($A = 1, \dots, 2n_H$) of positive chirality and $2n_H$ chiral spinors ζ_A of negative chirality. The spinors (ζ^A, ζ_A) of hypermultiplets are called the *hyperini* and the $4n_H$ real scalar fields q^X are the *hyperscalars*. The latter can be seen locally as coordinates of a scalar manifold M_H which is constrained by $\mathcal{N} = 2$ supersymmetry to be a quaternionic manifold [43].¹¹ It follows that locally, the $4n_H$ hyperscalar fields can be seen as n_H quaternions.¹²

If the $\mathcal{N} = 2$ supersymmetric theory admits an action, the scalar fields q^X define a sigma model with target space the hyperscalar manifold M_H which is required to be a Riemannian space with metric g_{XY} :

$$e^{-1} \mathcal{L}_{\text{hyper}} = -\frac{1}{2} g_{XY} \nabla_\mu q^X \nabla^\mu q^Y. \quad (2.22)$$

We will denote the quaternionic vielbein by f_X^{iA} . The metric g_{XY} is computed from the vielbein as

$$g_{XY} = f_X^{iA} f_{YiA}, \quad (2.23)$$

where $f_{YiA} = (f_Y^{iA})^*$.

Later, when considering the supersymmetry transformations we will also find the tensor f_{iA}^X , which is the inverse of f_X^{iA} as a $4n_H \times 4n_H$ matrix:

$$f_Y^{iA} f_{iA}^X = \delta_Y^X, \quad f_X^{iA} f_{jB}^X = \delta_j^i \delta_B^A. \quad (2.24)$$

The holonomy of the scalar manifold can be checked to be contained in $\text{Sp}(1) \otimes \text{Sp}(n_H)$. Thus, the scalar manifold of hypermultiplets is by definition a *quaternionic-Kähler manifold*.

We can define a triplet J^x ($x = 1, 2, 3$) of complex structures:

$$(J^x)_X^Y = i f_X^{iA} (\sigma^x)_i^j f_{jA}^Y \quad (2.25)$$

which satisfy the multiplication table of quaternionic units

$$J^x J^y = -\delta^{xy} \mathbf{1}_{4n_H} + \varepsilon^{xyz} J^z, \quad (2.26)$$

and are covariantly constant

$$\nabla J^x \equiv \nabla^{\text{LC}} J^x + 2\varepsilon^{xyz} \omega^y J^z = 0, \quad (2.27)$$

¹¹For a review of quaternionic geometry see [44, 45, 62]. In particular, we shall use the conventions of appendix B of [62].

¹²The set of quaternions is defined by $\mathbb{H} = \{q_0 1 + q_1 i + q_2 j + q_3 k | q_i \in \mathbb{R}\}$, with the elements of the basis satisfying i, j, k together with all cyclic permutations and $i^2 = j^2 = k^2 = -1$. A quaternion can be represented by a 2×2 matrix as the imaginary quaternions can be represented by $(i, j, k) = (-i\sigma^1, -i\sigma^2, -i\sigma^3)$ and therefore we can write $q = q^0 \mathbf{1} - iq^x \sigma^x$.

where ∇^{LC} is the Levi-Civita covariant derivative for the metric g_{XY} , and ω^x is an $\text{SU}(2)$ connection.

Any real linear combination $J = a_x J^x$ defines a complex structure ($J^2 = -\mathbf{1}$) if

$$\|\vec{a}\|^2 = (a_1)^2 + (a_2)^2 + (a_3)^2 = 1. \tag{2.28}$$

It follows that at each point of the manifold there is a sphere of complex structures. Two such complex structures are related by an $\text{SU}(2)$ rotation. A *quaternionic structure* is the space of all these complex structures $a_x J^x$. It is globally well defined although this is not necessary the case of an individual complex structure $J = a_x J^x$ as the $\text{SU}(2)$ bundle is non-trivial.

A quaternionic-Kähler manifold is an Einstein space and the quaternionic structure J_{XY}^x is proportional to the $\text{SU}(2)$ curvature $\mathcal{R}^x \equiv d\omega^x + \epsilon^{xyz}\omega^y\omega^z$:

$$R_{XY} = \frac{1}{4n_H} g_{XY} R, \quad \mathcal{R}_{XY}^x = \frac{1}{2} \nu J_{XY}^x, \quad \nu = \frac{1}{4n_H(n_H + 2)} R. \tag{2.29}$$

In $\mathcal{N} = 2$ supergravity, the constant ν is proportional to the gravity coupling constant $\nu = -\kappa^2$. We work in units in which $\kappa = 1$, that is $\nu = -1$.

In flat space, the supersymmetry generators are globally defined and therefore the $\text{SU}(2)$ bundle is trivial and the holonomy is contained in $\text{Sp}(n_H)$ and M_H is an *hyperkähler manifold*. An hyperKähler manifold can be seen as a quaternionic-Kähler manifold with a trivial $\text{Sp}(1)$ bundle so that the $\text{Sp}(1)$ -curvature vanishes $\mathcal{R}^x = 0$.

The $\text{Sp}(1) \simeq \text{SU}(2)$ and $\text{Sp}(n_H)$ bundles can be understood geometrically as follows: $\text{SU}(2)$ corresponds to the R-symmetry group that rotates the supersymmetry generators while $\text{Sp}(n_H)$ is the space of linear transformations of n_H hypermultiplets that preserve the metric of the scalar fields. It also acts on the hyperscalars q^X and the hyperini.

The non-triviality of the $\text{SU}(2) \simeq \text{Sp}(1)$ bundle in $\mathcal{N} = 2$ supergravity is responsible for the non-existence of FI terms but, ironically, it is also the main instrument to construct $\mathcal{N} = 2$ supergravity potentials that are compatible with $\mathcal{N} = 1$ supergravity with D -term endowed with constant FI term as we shall review shortly.

2.4 Isometries, gauging and scalar potential

In $\mathcal{N} = 2$ supergravity coupled to vector multiplets and hypermultiplets, the only way to generate a scalar potential is to promote some of the symmetries of the scalar manifold to be local symmetries. This implies a choice of the Killing vectors of the scalar manifolds and a choice of vector fields that will be used as gauge fields in the covariant derivatives.

In this paper we only consider Abelian gauging of the symmetries of M_H . The gauged symmetry is defined by the transformation with parameters α^I :

$$\delta_G q^X = -g\alpha^I k_I^X, \tag{2.30}$$

where k_I^X are the Killing vectors that we will gauge with the vector fields W_μ^I . To gauge a symmetry all the derivatives of the hyperscalars have to be extended to covariant derivatives. The gauge field is taken from the vector multiplets:

$$\nabla_\mu q^X = \partial_\mu q^X + gW_\mu^I k_I^X. \tag{2.31}$$

A gauging generates a deformation of the supersymmetry transformations by fermionic shifts S^{ij} , \mathcal{N}_{ij}^α and \mathcal{N}^{iA} :

$$\delta\psi_\mu^i = \dots - g\gamma_\mu S^{ij}\epsilon_j, \tag{2.32}$$

$$\delta\lambda_i^\alpha = \dots + gN_{ij}^\alpha\epsilon^j, \tag{2.33}$$

$$\delta\zeta^A = \dots + g\mathcal{N}^{iA}\epsilon_{ij}\epsilon^j. \tag{2.34}$$

These fermionic shifts also contribute to fermionic mass terms in the Lagrangian:

$$e^{-1}\mathcal{L}_{\psi\text{mass}} = -gS^{ij}\bar{\psi}_{\mu i}\gamma^{\mu\nu}\psi_{\nu j} + \frac{1}{2}gg_{\alpha\bar{\beta}}N_{ij}^\alpha\bar{\lambda}^{\bar{\beta}i}\gamma^\mu\psi_\mu^j + 2g\mathcal{N}^{iA}\epsilon_{ij}\bar{\zeta}_A\gamma^\mu\psi_\mu^j + \text{h.c.} \tag{2.35}$$

Finally, in order to preserve supersymmetry of the action, the previous mass terms should be balanced by the following scalar potential quadratic in the fermionic shifts:

$$g^{-2}\mathcal{V} = \frac{1}{2}g_{\alpha\bar{\beta}}N_{ij}^\alpha N^{\bar{\beta}ij} + 2\mathcal{N}^{iA}\mathcal{N}_{iA} - 6S^{ij}S_{ij}. \tag{2.36}$$

The invariance of the action under supersymmetric transformations implies that the fermionic shifts are given by:

$$S^{ij} \equiv -\mathcal{P}_I^{ij}X^I, \quad N_{ij}^\alpha \equiv \epsilon_{ij}k_I^\alpha\bar{X}^I - 2\mathcal{P}_{Iij}\bar{f}_\beta^I g^{\alpha\bar{\beta}}, \quad \mathcal{N}^{iA} \equiv -if_X^{iA}k_I^X\bar{X}^I, \tag{2.37}$$

where $\mathcal{P}_I^{ij} = \mathcal{P}_I^x(i\sigma^x)^{ij}$ and $\mathcal{P}_{Iij} = \mathcal{P}_I^x(i\sigma^x)_{ij}$ are the *moment maps* [44, 45, 62] related to the Killing vectors k_I^X of the quaternionic-Kähler manifold and k_I^α are the Killing vectors of the special manifold. Taking into account the relations (2.37), the scalar potential reads

$$\begin{aligned} g^{-2}\mathcal{V} &= \frac{1}{2}g_{\alpha\bar{\beta}}N_{ij}^\alpha N^{\bar{\beta}ij} + 2\mathcal{N}^{iA}\mathcal{N}_{iA} - 6S^{ij}S_{ij}, \\ &= (g_{\alpha\bar{\beta}}k_I^\alpha k_J^{\bar{\beta}} + 2g_{XY}k_I^X k_J^Y)\bar{X}^I X^J + 4(U^{IJ} - 3\bar{X}^I X^J)\mathcal{P}_I^x\mathcal{P}_J^x. \end{aligned} \tag{2.38}$$

We also have

$$U^{IJ} \equiv g^{\alpha\bar{\beta}}f_\alpha^I f_{\bar{\beta}}^J = -\frac{1}{2}(\text{Im } \mathcal{N})^{-1|IJ} - \bar{X}^I X^J, \quad f_\alpha^I \equiv \left(\partial_\alpha + \frac{1}{2}\partial_\alpha\mathcal{K}\right)X^I, \quad X^I = e^{\frac{\mathcal{K}}{2}}Z^I. \tag{2.39}$$

As the scalar fields of vector multiplets transform in the adjoint representation of the gauge group, the Killing vectors $k_J^\alpha, k^{\bar{\alpha}J}$ of the special manifold vanish for Abelian gauging. In particular, the sector $(4g_{\alpha\bar{\beta}}k_I^\alpha k_J^{\bar{\beta}}\bar{X}^I X^J)$ of the scalar potential is not present for Abelian gauging.

2.4.1 Moment map and Fayet-Iliopoulos terms in $\mathcal{N} = 2$ supergravity

The moment map \mathcal{P}_I^x appearing in the fermionic shifts and the scalar potentials is defined as a solution of the following equation [44, 45, 62]:

$$\frac{1}{2}k_I^X J_{XY}^x dq^Y = \nabla_Y \mathcal{P}_I^x, \quad \text{where } \nabla \mathcal{P}_I^x = d\mathcal{P}_I^x + 2\varepsilon^{xyz}\omega^y \mathcal{P}_I^z. \tag{2.40}$$

In rigid $\mathcal{N} = 2$ supersymmetry, the $SU(2)$ curvature of the quaternionic manifold is trivial. The equation defining the moment map is then

$$\frac{1}{2}k_I^X J_{XY}^x = \partial_Y \mathcal{P}_I^x. \tag{2.41}$$

In flat space, as the moment map is covered by a total derivative in its defining equation, it is determined only modulo an arbitrary triplet ξ^x of real constants:

$$\mathcal{P}_I^x \sim \mathcal{P}_I^x + \xi_I^x. \tag{2.42}$$

These constants ξ^x are the $\mathcal{N} = 2$ Fayet-Iliopoulos terms.

In local $\mathcal{N} = 2$ supersymmetry, due to the non-trivial $SU(2)$ connection, the triplet moment maps cannot be shifted by arbitrary constants and there are only Fayet-Iliopoulos terms when $n_H = 0$, that is when there are no hypermultiplets.

Indeed, as the $SU(2)$ bundle is non-trivial, the moment maps are uniquely defined by

$$4n_H \mathcal{P}_I^x = -J_{YZ}^x \nabla^Z k_I^Y, \tag{2.43}$$

thanks to the identity satisfied by any moment map \mathcal{P}_I^x [56]:

$$\nabla^u \nabla_u \mathcal{P}_I^x = 2n_H \mathcal{P}_I^x. \tag{2.44}$$

The uniqueness of \mathcal{P}_I^x implies in particular that the following equation

$$\nabla A^x = 0, \tag{2.45}$$

has no nontrivial solutions. Otherwise, $\mathcal{P}_I^x + A^x$ would be another solutions of equation (2.40). Moreover, if there is a non-trivial solution, the integrability condition $[\nabla_u, \nabla_v]A^x = 0$ implies that the $SU(2)$ curvature vanishes and therefore that the $SU(2)$ bundle is trivial. This is clearly not the case for a quaternionic-Kähler manifold.

The moment map can also be described in another way. A Killing vector preserves the connection ω^x and Kähler two forms J^x only modulo an $SU(2)$ rotation. Denoting by \mathcal{L}_I the Lie derivative with respect to k_I , we have

$$\mathcal{L}_I \omega^x = -\frac{1}{2} \nabla r_I^x, \tag{2.46}$$

or in terms of J^x

$$\mathcal{L}_I J^x = \varepsilon^{xyz} r_I^y J^z, \tag{2.47}$$

Here r_I^x is known as an $SU(2)$ *compensator*. The $SU(2)$ -bundle of a quaternionic manifold is non-trivial and therefore it is impossible to get rid of the compensator r_I^x by a redefinition of the $SU(2)$ connections.¹³ The moment map can be expressed in terms of the triplet of connections ω^x and the compensator r_I^x in the following way [44]:

$$\mathcal{P}_I^x = \frac{1}{2} r_I^x + \iota_I \omega^x, \tag{2.48}$$

where ι_I is an interior derivative with respect to k_I^X ($\iota_I \omega^x = k_I^Y \omega_Y^x$).

¹³Again, this is in contrast with $\mathcal{N} = 2$ rigid supersymmetry, since hyper-Kähler manifolds have a trivial $SU(2)$ bundle, and therefore no compensator.

2.5 $\mathcal{N} = 2$ Supersymmetry transformations

The supersymmetry transformations will be used later on to obtain the BPS equations for the string configuration.

The supersymmetry transformations involve the geometrical objects that we just discussed: the moment map, the Killing vectors and the metric of the scalar manifold.

The bosons transform as

$$\begin{aligned}
 \delta e_\mu^a &= \frac{1}{2} \bar{\epsilon}^i \gamma^a \psi_{\mu i} + \frac{1}{2} \bar{\epsilon}_i \gamma^a \psi_\mu^i, \\
 \delta W_\mu^I &= \frac{1}{2} (\mathcal{D}_\alpha X^I) \varepsilon^{ij} \bar{\epsilon}_i \gamma_\mu \lambda_j^\alpha + \frac{1}{2} (\mathcal{D}_{\bar{\alpha}} \bar{X}^I) \varepsilon_{ij} \bar{\epsilon}^i \gamma_\mu \lambda^{\bar{\alpha} j} + \varepsilon^{ij} \bar{\epsilon}_i \psi_{\mu j} X^I + \varepsilon_{ij} \bar{\epsilon}^i \psi_\mu^j \bar{X}^I, \\
 \delta z^\alpha &= \frac{1}{2} \bar{\epsilon}^i \lambda_i^\alpha, \\
 \delta q^X &= -i f_{iA}^X \bar{\epsilon}^i \zeta^A + i f^{XiA} \bar{\epsilon}_i \zeta_A.
 \end{aligned} \tag{2.49}$$

Here we have used the following definitions:

$$\varepsilon^{ij} = i\sigma^2 \quad \varepsilon_{ij} \varepsilon^{kj} = \delta_i^k \quad \text{and} \quad \bar{\epsilon}^i = (\epsilon^i)^\dagger \gamma_0 \tag{2.50}$$

For a bosonic configuration, the $N = 2$ supersymmetry transformations of the left-handed fermionic fields are:

$$\begin{aligned}
 \delta \psi_\mu^i &= \nabla_\mu(\omega) \epsilon^i + \frac{1}{4} \gamma^{\rho\sigma} T_{\rho\sigma}^- \varepsilon^{ij} \gamma_\mu \epsilon_j - g \gamma_\mu S^{ij} \epsilon_j, \\
 \delta \lambda_i^\alpha &= \nabla z^\alpha \epsilon_i - \frac{1}{2} g^{\alpha\bar{\beta}} \mathcal{D}_{\bar{\beta}} \bar{X}^I \text{Im} \mathcal{N}_{IJ} F_{\mu\nu}^- J \gamma^{\mu\nu} \varepsilon_{ij} \epsilon^j + g N_{ij}^\alpha \epsilon^j, \\
 \delta \zeta^A &= \frac{1}{2} i f_X^{Ai} \nabla q^X \epsilon_i + g \mathcal{N}^{iA} \varepsilon_{ij} \epsilon^j.
 \end{aligned} \tag{2.51}$$

The fermionic shifts (S^{ij} , N_{ij}^α and \mathcal{N}^{iA}) are given in equation (2.37). The associated scalar potential is presented in equation (2.38). The covariant derivatives are

$$\begin{aligned}
 \nabla_\mu(\omega) \epsilon^i &\equiv \left(\partial_\mu + \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} \right) \epsilon^i + \frac{1}{2} i A_\mu \epsilon^i + V_{\mu j}^i \epsilon^j, \\
 \nabla_\mu z^\alpha &= \partial_\mu z^\alpha + g W_\mu^I k_I^\alpha, \\
 \nabla_\mu q^X &= \partial_\mu q^X + g W_\mu^I k_I^X.
 \end{aligned} \tag{2.52}$$

We included here the effect of a gauging in the vector multiplet sector by the Killing vector k_I^α describing the transformations under the gauge symmetry of the vector multiplet scalar similar to the definition of k_I^X for the hypermultiplet scalars.

The $SU(2)$ connection $V_{\mu i}^j$ is related to the quaternionic-Kähler $SU(2)$ connection and gets a contribution from the moment map when isometries of the quaternionic-Kähler manifold have been gauged:

$$V_{\mu i}^j = \partial_\mu q^X \omega_{Xi}^j + g W_\mu^I \mathcal{P}_{Ii}^j. \tag{2.53}$$

A_μ are the components of the one-form gauge field of the Kähler $U(1)$:

$$A = -\frac{i}{2} (\partial_\alpha \mathcal{K} dz^\alpha - \partial_{\bar{\alpha}} \mathcal{K} d\bar{z}^{\bar{\alpha}}). \tag{2.54}$$

In the case of gauging in the vector multiplet sector, this is modified by a scalar moment map similar to the SU(2) connection. The dressed graviphoton is given by

$$T_{\mu\nu}^- = F_{\mu\nu}^{-I} \text{Im} \mathcal{N}_{IJ} X^J. \tag{2.55}$$

2.6 Consistent reduction of supersymmetry

It is often important to reduce the number of supersymmetries of a given theory. This is usually motivated by phenomenology as one would like in general to have the minimal amount of supersymmetry ($\mathcal{N} = 1$). There are many ways to realize a reduction of the number of supersymmetries of a given theory. One can break supersymmetry directly by introducing by hand supersymmetry breaking terms in the Lagrangian. Two other methods that appear naturally in the study of string inspired supergravity theories are *spontaneous supersymmetry breaking* and *consistent reduction of supersymmetry* [46–48].

In supergravity theories coming from string theory, spontaneous supersymmetry breakings arrive naturally from compactification with fluxes and/or torsion. In such a case, one ends up with gauged supergravity theories (or with a superpotential in $\mathcal{N} = 1$ supergravity) in which some of the gravitini become massive with masses coming from the fermionic shifts associated with the scalar potential. Consistent truncations, on the other hand do not require any fermionic masses, they consist of a reduction of the number of fields of the theory (including some massless fields) while respecting the equations of motion of the theory. It is then possible to get rid of some gravitini and supersymmetry generators. Although they might appear quite artificial at first look from a purely supergravity point of view, consistent reductions are naturally realized in string theory, for example by models containing orbifolds and/or orientifolds. Moreover some spontaneous supersymmetry breaking are also consistent reduction in the sense that all solutions of the equations of motion of the reduced theory are also solutions of the mother theory.

In a consistent truncation, any solution of the equations of motion of the reduced theory correspond to a solution of the mother theory. If $\{\phi\} = \{\phi^r\} \cup \{\phi^T\}$ are the fields of the mother theory \mathcal{L} , where $\{\phi^r\}$ are those that survive the truncation to the reduced theory \mathcal{L}^r and $\{\phi^T\}$ are the truncated field, the truncation ansatz is simply $\phi^T = 0$. A consistent truncation is such that the truncation ansatz commutes with the equations of motion

$$\text{Consistent truncation : } \frac{\delta}{\delta\phi} (\mathcal{L}|_{\phi^T=0}) = \left(\frac{\delta\mathcal{L}}{\delta\phi} \right) \Big|_{\phi^T=0}, \tag{2.56}$$

and therefore any solutions of the reduced theory is also a solution of the mother theory

$$\text{Consistent truncation : } \frac{\delta\mathcal{L}^r}{\delta\phi^r} = 0 \implies \frac{\delta\mathcal{L}}{\delta\phi} = 0. \tag{2.57}$$

It is possible that a spontaneous supersymmetry breaking implies a consistent truncation, but this is not necessarily the case.

2.7 $\mathcal{N} = 2 \implies \mathcal{N} = 1$ consistent truncations

Consistent truncation of supersymmetry in the context of supergravity is not a trivial task. A \mathcal{N} -extended supergravity theory admits \mathcal{N} -gravitini and cannot be seen in general as a special case of a $\mathcal{N}' < \mathcal{N}$ supergravity theory. Indeed the extra $(\mathcal{N} - \mathcal{N}')$ gravitini generate couplings that are inconsistent with the constraints of a \mathcal{N}' extended supergravity. Let us for example illustrate a few examples of incompatibilities between $\mathcal{N} = 2$ supergravity theory and $\mathcal{N} = 1$ supergravity:

- The graviphoton appearing in the supersymmetry transformation of the gravitini in $\mathcal{N} = 2$ supergravity. Such a term is inconsistent with the supersymmetry transformation of $\mathcal{N} = 1$ supergravity.
- In $\mathcal{N} = 2$ supergravity, n_H hypermultiplets cannot be seen as $2n_H$ chiral multiplets. Indeed, the scalar fields of n_H hypermultiplets define a quaternionic geometry and in general a quaternionic manifold is not even a complex manifold and therefore does not qualify to describe the geometry of $\mathcal{N} = 1$ chiral multiplets as the latter is supposed to be Kähler-Hodge.
- $\mathcal{N} = 2$ vector multiplets. Here the scalar fields of the vector multiplets define a special manifold. A special manifold is a Kähler-Hodge manifold and therefore one would expect that n_V $\mathcal{N} = 2$ vector multiplets can be seen as n_V gauge multiplets of $\mathcal{N} = 1$ supergravity together with n_V chiral multiplets. However, this is not in general the case. Indeed, the coupling matrix \mathcal{N}_{IJ} appearing in the kinetic terms of $\mathcal{N} = 2$ gauge fields is not restricted to be an holomorphic function whereas this is mandatory in $\mathcal{N} = 1$ supergravity coupled to gauge fields.¹⁴

The previous points illustrate some of the requirements that a consistent truncation of $\mathcal{N} = 2$ supergravity to $\mathcal{N} = 1$ supergravity should satisfy: the graviphoton should vanish, the quaternionic-Kähler manifold should reduce to a Kähler-Hodge manifold and the coupling matrix \mathcal{N} should be holomorphic after the truncation.

The conditions ensuring a consistent truncation in supergravity have been analyzed carefully in [46–48]. In particular we see that in a $\mathcal{N} = 2 \implies \mathcal{N} = 1$ consistent truncation, at least half of the degrees of freedom have to be truncated. In the gravity multiplet, the graviphoton and one gravitino have to be truncated. The quaternionic-Kähler manifold has to admit a Kähler-Hodge submanifold in order to satisfy the constraints of $\mathcal{N} = 1$ supergravity. A hypermultiplet is fully truncated or reduces to a unique chiral multiplet. A vector multiplet can be completely truncated or reduces to a gauge or a chiral multiplet.

2.7.1 A dictionary of $\mathcal{N} = 2 \implies \mathcal{N} = 1$ consistent truncations

We shall only consider $\mathcal{N} = 2 \implies \mathcal{N} = 1$ consistent truncation in which we keep the first gravitino but truncate the second one, that is

$$\psi_{\mu 2} = \psi_{\mu}^2 = \epsilon_2 = \epsilon^2 = 0. \tag{2.58}$$

¹⁴The same argument shows that even in rigid supersymmetry, $\mathcal{N} = 2$ and $\mathcal{N} = 1$ supersymmetry are in general not compatible.

After a consistent truncation, the resulting $\mathcal{N} = 1$ supergravity is such that

$$\epsilon_L = \epsilon^1, \quad \psi_{\mu L} = \psi_\mu^1, \quad (2.59a)$$

$$D^\alpha = -2g \operatorname{Im} \mathcal{N}^{-1|\alpha\beta} (\mathcal{P}_\beta^0 + \mathcal{P}_\beta^3), \quad e^{\frac{1}{2}K} W = -2g X^\beta (P_\beta^1 - i\mathcal{P}_\beta^2), \quad (2.59b)$$

$$\lambda_L^\alpha = -\lambda_2^\beta \mathcal{D}_\beta X^\alpha, \quad f_{\alpha\beta} = i\mathcal{N}_{\alpha\beta}. \quad (2.59c)$$

To write these equations we have decomposed the $\mathcal{N} = 2$ vector indices I as $I \rightarrow (\alpha, \tilde{\alpha})$, with $\alpha = 1, \dots, n_G$ running over the n_G gauge multiplets that survive the truncation to the $\mathcal{N} = 1$ theory, and $\tilde{\alpha} = 0, n_G + 1, \dots, n_V$ over the complementary indices, which label the chiral multiplets of the truncated theory coming from $\mathcal{N} = 2$ vector multiplets. Here D^α , W and $f_{\alpha\beta}$ are the D -term the superpotential and the kinetic functions of the reduced $\mathcal{N} = 1$ theory respectively. λ_L^α is the fermionic partner of the surviving gauge field after the truncation.

Among the conditions for a consistent truncation let us mention the following

$$T_{\mu\nu} = 0, \quad \omega_X^1 = \omega_X^2 = 0, \quad (2.60)$$

where $T_{\mu\nu}$ is the graviphoton and ω_X^x is the $SU(2)$ -connection of the quaternionic-Kähler manifold.

The scalar manifold of the reduced $\mathcal{N} = 1$ theory is a direct product $M_V^{KH} \otimes M_H^{KH}$ where M_V^{KH} is the reduced manifold coming from the scalar manifold M_V of vector multiplets and M_H^{KH} is a Kähler-Hodge submanifold of the quaternionic-Kähler manifold M_H :

$$M = M_V \otimes M_H \implies M_{KH} = M_V^{KH} \otimes M_H^{KH}. \quad (2.61)$$

The $U(1)$ connection of M_H^{KH} is determined by ω_X^3 .

2.8 $\mathcal{N} = 1$ FI terms from a $\mathcal{N} = 2$ consistent truncation

As discussed in the introduction, in the context of $\mathcal{N} = 1$ supergravity, constant Fayet-Iliopoulos terms play an important role in cosmology and in the study of topological defects and in particular for constructing potentials admitting cosmic string solutions of Nielsen-Olesen type.

However as $\mathcal{N} = 2$ supergravity theories do not admit Fayet-Iliopoulos terms in the presence of hypermultiplets, it is not clear how these constant FI terms of $\mathcal{N} = 1$ supergravity could be obtained from $\mathcal{N} = 2$ supergravity. Indeed, as we have discussed in the previous section, $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supergravity theories are not necessary compatible. However, starting from an $\mathcal{N} = 2$ supergravity theory one can go to $\mathcal{N} = 1$ by spontaneous supersymmetry breaking or by a consistent truncation.

If we consider a consistent truncation in which we truncate the second gravitino, we see from the dictionary of the previous section that a constant FI term can be generated by a moment map with a constant \mathcal{P}^3 . This would require defining a gauging of a symmetry of the quaternionic-Kähler manifold.

Consider a Killing vector k_1 of a quaternionic-Kähler manifold M_H such that

$$\mathcal{L}_{k_1} J^x = \varepsilon^{xyz} r_{k_1}^y J^z, \quad (2.62)$$

with $r_{k_1}^x = \delta_3^x$. The moment is then

$$\mathcal{P}_{k_1}^x = \iota_{k_1} \omega^x + \frac{1}{2} \delta_3^x, \quad (2.63)$$

if we gauge ξ^{k_1} and can have $\iota_{k_1} \omega^x = 0$ we have a candidate $\mathcal{N} = 2$ gauging that can be reduced to an $\mathcal{N} = 1$ supergravity with a vanishing superpotential, and a D -term endowed with a constant FI terms because D -term is proportional to $\mathcal{P}_{k_1}^3$ since there is no gauging along the special manifold:

$$\mathcal{D} \sim \xi \quad (2.64)$$

Such a constant FI term that we obtain in the reduced $\mathcal{N} = 1$ theory is coming from the $SU(2)$ compensator $r_{k_1}^x$ which is due to the non-triviality of the $SU(2)$ -bundle. Ironically it is the very same property that forbids the introduction of $\mathcal{N} = 2$ FI terms in supergravity coupled to hypermultiplets!

The mechanism that we have just discussed was first presented in [18] in a model where $\mathcal{N} = 2$ supergravity is coupled to one vector multiplet and one chiral multiplet. After truncation it yields a $\mathcal{N} = 1$ supergravity theory coupled to a gauge and a chiral multiplet and admitting a D -term potential endowed with a constant FI term. The FI term was used to construct the first example of a half-BPS cosmic string solutions of $\mathcal{N} = 2$ supergravity.

Using the property of consistent truncations, the construction of [18] can be described as the embedding of a $\mathcal{N} = 1$ half-BPS D -term cosmic string solution in a $\mathcal{N} = 2$ supergravity theory. Indeed, any solution of the reduced $\mathcal{N} = 1$ theory is also a solution of the mother $\mathcal{N} = 2$ theory. One can alternatively consider the use of consistent truncation in [18] as a trick to simplify the $\mathcal{N} = 2$ BPS equations so that they look similar to those of a $\mathcal{N} = 1$ supergravity model.

2.8.1 A simple example

We shall now present a realization of the mechanism discussed above in a simple example. In this example, we shall focus on the hypermultiplets. The rest of the paper will emphasize the role of the special geometry. For the quaternionic-Kähler manifold, we take a quaternionic space of quaternionic dimension one:

$$\frac{SO(4, 1)}{SO(4)} = \frac{Sp(1, 1)}{Sp(1) \cdot Sp(1)}, \quad (2.65)$$

with real coordinates $(b^0 = e^h, b^1, b^2, b^3)$ and admitting the metric

$$ds^2 = \frac{1}{(b^0)^2} \left[(db^0)^2 + |\vec{db}|^2 \right]. \quad (2.66)$$

The vielbein and $SU(2)$ connection are given by

$$f = \frac{1}{\sqrt{2} b^0} (db^0 \mathbf{1}_2 + i \vec{db} \cdot \vec{\sigma}), \quad \vec{\omega} = -\frac{d\vec{b}}{b^0}. \quad (2.67)$$

The isometry

$$k_1 = 2b^1 \frac{\partial}{\partial b^2} - 2b^2 \frac{\partial}{\partial b^1}, \quad (2.68)$$

rotates the connection ω^x as

$$\mathcal{L}_{k_1} \omega^x = 2\varepsilon^{xyz} \omega^z \delta_3^y = -\nabla \delta_3^x, \quad (2.69)$$

from which we identify the SU(2)-compensator $r_{k_1}^y = 2\delta_3^x$. We can then compute the moment map

$$\mathcal{P}_{k_1} = \iota_{k_1} \omega + \frac{1}{2} r_{k_1} = \frac{1}{b^0} \begin{pmatrix} b^2 \\ -b^1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (2.70)$$

Along the quaternionic-Kähler manifold, the extremum of the scalar potential is at

$$b^1 = b^2 = 0, \quad (2.71)$$

leaving the constant moment map

$$\mathcal{P}_{k_1}|_{b^1=b^2=0} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (2.72)$$

The quaternionic-Kähler manifold $\frac{\text{SO}(4,1)}{\text{SO}(4)}$ reduces to the submanifold $\frac{\text{SO}(2,1)}{\text{SO}(2)}$

$$\frac{\text{SO}(4,1)}{\text{SO}(4)} \implies \frac{\text{SO}(2,1)}{\text{SO}(2)}, \quad (2.73)$$

with metric

$$ds^2 = \frac{1}{(b^0)^2} [(b^0)^2 + (b^3)^2]. \quad (2.74)$$

Defining $\phi = b^3 - ib^0$ we have a Kähler-Hodge metric

$$ds^2 = \frac{d\phi d\bar{\phi}}{(\text{Im } \phi)^2}, \quad (2.75)$$

admitting the Kähler potential $\mathcal{K} = -2 \log[i(\phi - \bar{\phi})]$.

We see that the quaternionic-Kähler manifold $\frac{\text{SO}(4,1)}{\text{SO}(4)}$ (which is not even a complex manifold) can be consistently reduced to a Kähler-Hodge manifold $\frac{\text{SO}(2,1)}{\text{SO}(2)}$.

In the present case, if we gauge

$$k = \xi k_1, \quad (2.76)$$

where k_1 is given in equation (2.68) we obtain after truncation ($b^1 = b^2 = 0$) an $\mathcal{N} = 1$ supergravity with vanishing superpotential and a D -term endowed with a constant FI term, using (2.59) we have:

$$D = -2g\mathcal{P}^3 = -2g\xi. \quad (2.77)$$

As we see the coefficient ξ in (2.76) determined the value of the constant FI term. This is because k_1 has been normalized such that the third component of the moment map is equal to one ($\mathcal{P}_{k_1}^3 = 1$). Since the Killing vector k_1 vanishes identically, there is no gauge symmetry in the reduced $\mathcal{N} = 1$ theory. But if we gauge a linear combination $k = \xi k_1 + k_2$ such that k_2 acting on b^0 and b^3 we can end up with an $\mathcal{N} = 1$ theory with an Abelian gauging and a constant Fayet-Iliopoulos term as we shall discuss in the next section.

2.8.2 Engineering cosmic strings in $\mathcal{N} = 2$ supergravity

Once we can engineer a gauging in $\mathcal{N} = 2$ supergravity yielding FI terms in a reduced $\mathcal{N} = 1$ supergravity, it is not difficult to identify a gauging that generates a $\mathcal{N} = 2$ potential admitting cosmic string solutions of the Nielsen-Olesen type. What we need is to have a $U(1)$ compact symmetry surviving the reduction and gauging the Higgs field of the string.

Since we work with a consistent truncation, if the reduced theory admits cosmic string solutions, they would also be solutions of the mother $\mathcal{N} = 2$ supergravity equations of motion. In this spirit, we first identify a Killing vector in the reduced $\mathcal{N} = 1$ theory which generates a compact $U(1)$ symmetry. We then lift up that symmetry to the quaternionic-Kähler manifold.¹⁵

Let us illustrate the point using the previous example. The reduced Kähler-Hodge manifold $\frac{SO(2,1)}{SO(2)}$ admits a unique compact symmetry

$$\delta\phi = -g(\phi^2 + 1). \tag{2.78}$$

The cosmic string solutions are based on the D -term potential

$$D \propto -2g \left(\frac{|\phi|^2 + 1}{\text{Im } \phi} - \xi \right). \tag{2.79}$$

The vacuum is a circle if we take $(\xi > 2)$ ¹⁶

$$D = 0 \implies (b^3)^2 + (b^0 - \frac{\xi}{2})^2 = \frac{\xi^2 - 4}{2}. \tag{2.80}$$

We then gauge in the mother $\mathcal{N} = 2$ supergravity theory the Killing vector

$$k = \xi k_1 + k_2, \tag{2.81}$$

where

$$k_2 = (\phi^2 + 1) \frac{\partial}{\partial \phi} + \dots \tag{2.82}$$

is an uplift of the compact symmetry of $\frac{SO(2,1)}{SO(2)}$ to the quaternionic-Kähler manifold $\frac{SO(4,1)}{SO(4)}$: the dots (\dots) stand for terms that vanish in the reduction.

In a sense, we can say that the method that we have presented here is a consistent embedding of an $\mathcal{N} = 1$ theory with a D -term endowed with a constant FI term into a gauged $\mathcal{N} = 2$ supergravity. The cosmic string solutions of the reduced theory are also valid solutions of the mother theory. Thus we also have a consistent embedding of a cosmic string solution of a $\mathcal{N} = 1$ supergravity theory into $\mathcal{N} = 2$ supergravity.

We note that the gauge symmetry of the reduced symmetry admits a unique fixed point

$$\phi = i, \tag{2.83}$$

¹⁵This is always possible as the sector of the scalar manifold of the $\mathcal{N} = 1$ coming from the quaternionic-Kähler manifold is a completely geodesic submanifold of the former. Therefore any symmetry of the reduced manifold can be lifted up to a symmetry of the mother manifold.

¹⁶ $\xi < 2$ is excluded as $b^0 = e^h > 0$.

where the D -term is such that

$$D \propto 2g(\xi - 2). \tag{2.84}$$

We shall see later on that the deficit angle of the cosmic string is proportional to $(\xi - 2)$.

3. The model

In this section we describe the model. We shall gauge a unique Abelian Killing vector of the quaternionic manifold. As for any Abelian gauging in $\mathcal{N} = 2$ supergravity, the scalars of the vector multiplet are neutral as they transform in the adjoint representation of the gauged group.

We will consider a simple model involving the minimum number of scalar fields required to have a cosmic string with a $\mathcal{N} = 2$ potential bounded from below using the $ST[2, n]$ manifold. As we shall see this will require $n \geq 1$ as the two first lines of the symplectic section are related to the graviphoton and for gauging of the graviphoton the scalar potential is not guaranteed to be bounded from below. Moreover, we would like to define a string configuration in $\mathcal{N} = 2$ supergravity which is compatible with $\mathcal{N} = 1$ supergravity. In such a case, the graviphoton has to decouple on the string configuration.

On the quaternionic side, we take the scalar manifold to be $\frac{SO(4,1)}{SO(4)}$. The same solution can be constructed using any normal quaternionic manifold. However for keeping the geometry as simple as possible we shall restrict ourselves to $\frac{SO(4,1)}{SO(4)}$. This will also enable us to compare our results with those of [18].

The choice of the Killing vector is a crucial part of the construction. For the specific quaternionic geometry that we consider it is explained in [18]. The treatment for a generic homogeneous quaternionic manifold will be presented in [59].

3.1 The very special Kähler manifold $\frac{SU(1,1)}{U(1)} \times \frac{SO(2,n)}{SO(2) \times SO(n)}$ in the Calabi-Vesentini section

We consider the Kähler-Hodge manifold

$$ST[2, 2 + n] = \frac{SU(1, 1)}{U(1)} \times \frac{SO(2, 2 + n)}{SO(2) \times SO(2 + n)}, \tag{3.1}$$

We will work in the Calabi-Vesentini basis defined by the holomorphic section:

$$v = \begin{pmatrix} Z^I \\ F_I \end{pmatrix}, \quad \text{with} \quad Z^I = \begin{pmatrix} \frac{1}{2}(1 + y^2) \\ i\frac{1}{2}(1 - y^2) \\ y^a \end{pmatrix}, \quad \text{and} \quad F_I = S\eta_{IJ}Z^J, \tag{3.2}$$

where $a = 1, \dots, n$ and $y^2 = y^a y^a$ and $\eta_{IJ} = \begin{pmatrix} \mathbf{1}_2 & \\ & -\mathbf{1}_n \end{pmatrix}$. The fields S and y^a parametrize respectively the manifold $\frac{SU(1,1)}{U(1)}$ and $\frac{SO(2,n)}{SO(2) \times SO(n)}$. The Calabi-Vesentini basis does not admit a prepotential, but can be rotated to a symplectic section which can be obtained from the cubic prepotential

$$F(S, y) = -\frac{1}{2}S y^a y^a. \tag{3.3}$$

The kähler potential for the Calabi-Vesentini section is

$$\mathcal{K} = -\log [i(S - \bar{S})] - \log \left[\frac{1}{2} (1 - 2\bar{y}^a y^a + |y^a y^a|^2) \right], \quad (3.4)$$

and the coupling matrix of the vector field

$$\mathcal{N}_{IJ} = (S - \bar{S}) \frac{Z_I \bar{Z}_J + \bar{Z}_I Z_J}{\bar{Z}^T \eta Z} + \bar{S} \eta_{IJ}. \quad (3.5)$$

The metric is given as usual by the second derivative of the Kähler potential

$$g_{S\bar{S}} = \frac{1}{(2 \operatorname{Im} S)^2}, \quad g_{b\bar{c}} = 2 \frac{(\delta^{b\bar{c}} - 2y^b \bar{y}^{\bar{c}})}{1 - 2\bar{y}^a y^a + |y^a y^a|^2} + 4 \frac{[\bar{y}^b - y^b (\bar{y}^{\bar{a}} \bar{y}^{\bar{a}})] [y^{\bar{c}} - \bar{y}^{\bar{c}} (y^a y^a)]}{(1 - 2\bar{y}^a y^a - |y^a y^a|^2)^2} \quad (3.6)$$

In the example we shall consider in this paper, we will restrict ourselves to the case $n = 1$, as it requires the minimum amount of fields: two complex scalar fields S and y and three vector fields $W_0, W_1, W \equiv W_2$. This specific case is immediately generalized to any n . In this case the metric for the scalar manifold M_V and the coupling matrix \mathcal{N} simplify to:

$$g_{S\bar{S}} = \frac{1}{(2 \operatorname{Im} S)^2}, \quad g_{y\bar{y}} = \frac{2}{(1 - y\bar{y})^2} \quad \mathcal{N} = \begin{pmatrix} S \mathbf{1}_2 & \\ & -\bar{S} \end{pmatrix} \quad (3.7)$$

3.2 The quaternionic manifold $\frac{\operatorname{SO}(4,1)}{\operatorname{SO}(4)}$

The quaternionic manifold of quaternionic dimension one

$$\frac{\operatorname{SO}(4,1)}{\operatorname{SO}(4)}, \quad (3.8)$$

has a very simple quaternionic structure which can be derived from the vielbein

$$f = \frac{1}{\sqrt{2}} \left(dh \mathbf{1} + i e^{-h} db^x \sigma^x \right), \quad (3.9)$$

where $x = 1, 2, 3$ and h, b^x are real fields. Its metric and $\operatorname{SU}(2)$ -connection are respectively

$$ds^2 = (dh)^2 + e^{-2h} [(db_1)^2 + (db_2)^2 + (db_3)^2], \quad (3.10)$$

$$\omega^x = -\frac{1}{2} e^{-h} db^x. \quad (3.11)$$

3.3 Killing vector and moment map

We will consider the same Abelian gauging as in [18]. With our choice of special geometry, on the submanifold $y = 0$, the graviphoton, (2.55), depends only on W_0 and W_1 . As we would like to put the graviphoton to zero on the string configuration we shall gauge the vector W_μ^2 .

The $\operatorname{U}(1)$ symmetry that we gauge is [18]:

$$k = \begin{pmatrix} 4b_3 \\ 4b_1 b_3 \\ 4b_2 b_3 \\ 2 [b_3^2 - e^{2h} + 1 - b_1^2 - b_2^2] \end{pmatrix} + \xi \begin{pmatrix} 0 \\ -2b_2 \\ 2b_1 \\ 0 \end{pmatrix}, \quad (3.12)$$

where we have arranged the tangent vector in the order $(\frac{\partial}{\partial h}, \frac{\partial}{\partial b_1}, \frac{\partial}{\partial b_2}, \frac{\partial}{\partial b_3})$. Although the previous Killing vector seems complicated at first sight it is defined in a precise and simple way on any symmetric normal quaternionic manifold using a solvable parametrization of the quaternionic manifold [59]. The moment map corresponding to k reads

$$\mathcal{P}^x = \begin{pmatrix} -2b_2 - 2b_1b_3e^{-h} \\ 2b_1 - 2b_2b_3e^{-h} \\ -e^{-h} [(b_3)^2 + 1 - (b_1)^2 - (b_2)^2] - e^h \end{pmatrix} + \xi \begin{pmatrix} e^{-h} b_2 \\ -e^{-h} b_1 \\ 1 \end{pmatrix}. \quad (3.13)$$

3.4 The scalar potential

In the Calabi-Vesentini symplectic section we have for any n :

$$U^{IJ} - 3\bar{X}^I X^J = -\frac{1}{i(S - \bar{S})} \eta^{IJ}. \quad (3.14)$$

Since $ImS < 0$, it follows that the scalar potential is always positive and bounded from below in the Calabi-Vesentini basis, provided that we gauge the vector W_I with η_{II} negative. This corresponds to the gauge field associated with the coordinate \mathbf{y} .

We shall gauge a unique Killing vector k^X of the hypermanifold. With our choice the scalar potential is then

$$\mathcal{V} = 4e^{-\rho} k^2 \frac{y\bar{y}}{(1 - y\bar{y})^2} + 2e^{-\rho} \mathcal{P}^x \mathcal{P}^x, \quad (3.15)$$

where $k^2 = g_{XY} k^X k^Y$.

The previous scalar potential has the following properties:

1. *The scalar potential \mathcal{V} is bounded from below*

$$\mathcal{V} \geq 0, \quad (3.16)$$

this is in sharp contrast to the case of the minimal special geometry where the scalar potential was not bounded from below and could be positive, negative or vanish depending on the value of the scalar fields.

2. *$y = 0$ is a critical point of the scalar potential \mathcal{V} :*

$$\frac{\partial \mathcal{V}}{\partial y} \Big|_{y=0} = 0. \quad (3.17)$$

3. *The scalar potential \mathcal{V} has a runaway behaviour in the dilaton field ρ :*

$$\mathcal{V} \propto e^{-\rho}. \quad (3.18)$$

4. A half-BPS cosmic string solution

In this section we will present a 1/2-BPS cosmic string solution of the full $\mathcal{N} = 2$ supergravity action. First we shall study the Minkowski vacua of the scalar potential. Next we will specify the field configuration characterizing the consistent reduction, and finally we will compute the BPS equations for the string configuration and analyze some of the string properties.

Minkowski vacua		
No Minkowski vacuum		$\xi \leq 0$
$y = 0,$	$\Phi = \frac{\xi}{2}i,$	$ \tilde{\Phi} ^2 = 1 - \frac{\xi^2}{4}$
	$\Phi = i,$	$\tilde{\Phi} = 0$
$y = 0,$	$\frac{ \Phi ^2+1}{\text{Im } \Phi} = \xi,$	$\tilde{\Phi} = 0$
		$2 < \xi$

Table 3: Type of vacua of the scalar potential for different values of the parameter ξ . Non-singular cosmic string solutions are only possible for $2 < \xi$.

4.1 Minkowski vacua of the scalar potential

The scalar potential is a sum of squares. It is easy to compute all its Minkowski vacua by looking at the zeroes of the different terms:

$$\mathcal{V} = 0 \implies (k = 0 \text{ or } y = 0) \text{ and } \mathcal{P}^x \mathcal{P}^x = \mathbf{0}. \tag{4.1}$$

To study the scalar potential, it is useful to introduce the following definitions:

$$\Phi = -b_3 + ie^h, \quad \tilde{\Phi} = b_1 + ib_2. \tag{4.2}$$

which yield

$$\mathcal{P}^x \mathcal{P}^x = \frac{4}{(\text{Im } \Phi)^2} \left| \Phi - \frac{\xi}{2}i \right|^2 |\tilde{\Phi}|^2 + \left(\frac{|\tilde{\Phi}|^2}{\text{Im } \Phi} - \frac{|\Phi|^2 + 1}{\text{Im } \Phi} + \xi \right)^2 \tag{4.3}$$

$$\tag{4.4}$$

It is then easy to find

$$\mathcal{P}^x \mathcal{P}^x = 0 \implies \begin{cases} \text{Case I} & : \Phi = \frac{\xi}{2}i, \quad |\tilde{\Phi}|^2 = 1 - \frac{\xi^2}{4}, \quad (0 < \xi < 2), \\ \text{Case II} & : \frac{|\Phi|^2+1}{\text{Im } \Phi} = \xi, \quad \tilde{\Phi} = 0, \quad (2 \leq \xi). \end{cases} \tag{4.5}$$

$k = 0$ has a unique solution given by the origin of the quaternionic manifold:

$$k = 0 \implies b_3 = b_2 = b_1 = h = 0 \iff \Phi = i \quad \text{and} \quad \tilde{\Phi} = 0. \tag{4.6}$$

Putting things together we have a Minkowski vacuum for each value of $0 < \xi$:

When $\xi \leq 0$ there are no Minkowski vacua. This implies in particular that for a gauging with $\xi \leq 0$, all the extrema of the potential are de Sitter vacua. However, the potential will not have an absolute minimum (for finite values of the fields) because of its runaway behaviour in the dilaton (3.18).

We shall use table 3 to explain our choice for the cosmic string configuration. In order to have a cosmic string solution we need to have a circle in the vacuum manifold. If we want the string configuration to be compatible with a consistent reduction of supersymmetry,

we shall have to truncate some of the scalar fields of the quaternionic manifold to end up with a Kähler-Hodge submanifold which is completely geodesic.

The appropriate choice of gauging to construct a cosmic string of the Nielsen-Olesen type is $\xi > 2$. Indeed, in that case, the vacuum is a circle defined by $\frac{|\Phi|^2+1}{\text{Im}\Phi} = \xi$. The Higgs field of the cosmic string is Φ . We shall keep $y = \tilde{\Phi} = 0$ not only in the vacuum but for all the string solutions in order to have a consistent truncation.

In the case $0 < \xi < 2$, we also have a circle in the vacuum. However, $\Phi = \frac{\xi}{2}i$ does not define a consistent truncation of the quaternionic manifold. To see this note that a gauge transformation 2.30 with the killing vector k given by (3.12) does not respect this condition for every value of $\tilde{\Phi}$.

In the case where $\xi = 2$, the vacuum is just a point and therefore there is no room for a cosmic string solution of the Nielsen-Olesen type.

4.2 Consistent truncation

The set of conditions that we impose on the bosonic fields defining the consistent reduction are:

$$\text{Consistent reduction ansatz: } \begin{cases} y & = 0, \\ \tilde{\Phi} & = 0, \\ W_0 = W_1 & = 0. \end{cases} \quad (4.7)$$

The condition $y = \tilde{\Phi} = 0$ was explained in the previous section. The conditions $W_0 = W_1 = 0$ ensure that the graviphoton (see equation (2.55)) vanishes as it should be in a consistent truncation to $\mathcal{N} = 1$ supergravity. Indeed, the graviphoton appears in the supersymmetry transformations of the gravitini in $\mathcal{N} = 2$ supergravity (2.51) but is absent in those of the gravitino of $\mathcal{N} = 1$ supergravity.

In this field configuration the quaternionic and special Kähler manifold reduce as follow:

$$\mathcal{M}_{SK} \times \mathcal{M}_Q \xrightarrow{y=\tilde{\Phi}=0} \left(\frac{\text{SU}(1,1)}{\text{U}(1)} \right)_S \times \left(\frac{\text{SU}(1,1)}{\text{U}(1)} \right)_\Phi \simeq \left(\frac{\text{SO}(2,2)}{\text{SO}(2) \times \text{SO}(2)} \right)_{S,\Phi}, \quad (4.8)$$

with Kähler potential

$$\mathcal{K} = -\log[-i(S - \bar{S})] - 2\log[-i(\Phi - \bar{\Phi})]. \quad (4.9)$$

Here S is an axion-dilaton field and Φ is the scalar field whose Higgs mechanism generates the cosmic string. Once we impose the condition $\tilde{\Phi} = 0$, the Killing vector of the quaternionic manifold that we have gauged only acts on Φ as:

$$\delta\Phi = -2g(\Phi^2 + 1), \quad (4.10)$$

which corresponds to the compact $\text{U}(1)$ symmetry of $\left(\frac{\text{SU}(1,1)}{\text{U}(1)} \right)_\Phi$.

4.3 Truncated $\mathcal{N} = 1$ Lagrangian and supersymmetry transformations

The bosonic sector of the $\mathcal{N} = 2$ supergravity action is:

$$\begin{aligned}
 e^{-1}\mathcal{L} = & \frac{1}{2}R + \frac{1}{4}(\text{Im } \mathcal{N})_{IJ}F^{I\mu\nu}F_{\mu\nu}^J + \frac{e^{-1}}{8}(\text{Re } \mathcal{N})_{IJ}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}^IF_{\rho\sigma}^J \\
 & - \frac{1}{(2\text{Im } S)^2}\partial_\mu S\partial^\mu \bar{S} - \frac{2}{(1-y\bar{y})^2}\partial_\mu y\partial^\mu \bar{y} \\
 & - \frac{1}{2}g_{XY}\nabla_\mu q^X\nabla^\mu q^Y + \frac{2g^2}{\text{Im } S}\left[2k^2\frac{y\bar{y}}{(1-y\bar{y})^2} + \mathcal{P}^x\mathcal{P}^x\right]^2, \quad (4.11)
 \end{aligned}$$

where:

$$(\text{Im } \mathcal{N})_{IJ} = \text{Im } S \mathbf{1}_3 \quad (\text{Re } \mathcal{N})_{IJ} = \begin{pmatrix} \text{Re } S \mathbf{1}_2 & \\ & -\text{Re } S \end{pmatrix} \quad (4.12)$$

and the metric g_{XY} is given by (3.10). The hyperscalars are organized as $q^X = (h, \vec{b})$. The covariant derivatives are defined by (2.31) and the killing vector (3.12) is gauged by W^2 . The square of the moment map, $\mathcal{P}^x\mathcal{P}^x$ is given in (4.4).

After setting to zero the truncated fields (4.7) and imposing $\text{Re}(S) = 0$ the bosonic sector of the $\mathcal{N} = 1$ reduced action reads:

$$\begin{aligned}
 e^{-1}\mathcal{L} = & \frac{1}{2}R + \frac{\text{Im } S}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{+} \frac{e^{-1}}{8}\text{Re } S\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \\
 & - \frac{1}{(2\text{Im } S)^2}\partial_\mu S\partial^\mu \bar{S} - \frac{1}{2(\text{Im } \Phi)^2}\nabla_\mu \Phi\nabla^\mu \bar{\Phi} - 2\frac{g^2}{\text{Im } S}\left[\frac{|\Phi|^2 + 1}{\text{Im } \Phi} - \xi\right]^2, \quad (4.13)
 \end{aligned}$$

where

$$\nabla_\mu \Phi = \partial_\mu \Phi - 2gW_\mu(\Phi^2 + 1). \quad (4.14)$$

In the truncated theory the following relations hold:

$$S^{ij} = T_{\mu\nu}^- = N_{ij}^S = \mathcal{N}^{iA} = 0, \quad (4.15a)$$

$$e^{\mathcal{K}} = \frac{1}{-\text{Im } S}, \quad (4.15b)$$

$$\mathcal{D}_S X^I = \frac{e^{\frac{3}{2}\mathcal{K}}}{4} \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}, \quad \mathcal{D}_{y^a} X^I = e^{\mathcal{K}/2}\delta_a^I, \quad (4.15c)$$

$$V_i{}^j = i(\omega^3 + gW\mathcal{P}^3)_i{}^j, \quad (4.15d)$$

$$N_{ij}^y = -e^{\mathcal{K}/2}\mathcal{P}_{ij}^3, \quad (4.15e)$$

$$A = -\frac{i}{4}\frac{dS + d\bar{S}}{\text{Im } S}. \quad (4.15f)$$

The previous relations are useful to compute the supersymmetry transformations of

the fermions from equation (2.51):

$$\begin{aligned}
 \delta\psi_\mu^1 &= \left(\partial_\mu + \frac{1}{4}\omega_{\mu|ab}\gamma^{ab} + \frac{1}{2}iA_\mu + \frac{1}{2}iA_\mu^B \right) \epsilon^1, & \delta\psi_\mu^2 &= \left(\partial_\mu + \frac{1}{4}\omega_{\mu|ab}\gamma^{ab} + \frac{1}{2}iA_\mu - \frac{1}{2}iA_\mu^B \right) \epsilon^2, \\
 \delta\lambda_2^y &= -\frac{e^{\frac{\kappa}{2}}}{2}(\text{Im } S)F_{12}\gamma^{12}\epsilon^1 - ig e^{\frac{\kappa}{2}}\mathcal{P}^3 \epsilon^1, & \delta\lambda_1^y &= \frac{e^{\frac{\kappa}{2}}}{2}(\text{Im } S)F_{12}\gamma^{12}\epsilon^2 - ig e^{\frac{\kappa}{2}}\mathcal{P}^3 \epsilon^2, \\
 \delta\lambda_1^S &= \not{\nabla} S \epsilon_1, & \delta\lambda_2^S &= \not{\nabla} S \epsilon_2, \\
 \delta\zeta^1 &= \frac{\sqrt{2}}{4\text{Im } \Phi} \not{\nabla} \Phi \epsilon_1, & \delta\zeta^2 &= -\frac{\sqrt{2}}{4\text{Im } \Phi} \not{\nabla} \bar{\Phi} \epsilon_2.
 \end{aligned} \tag{4.16}$$

In these equations A_μ^B is the quaternionic matter connection of the gravitini:

$$A_\mu^B = 2\omega_\mu^3 + 2gW_\mu\mathcal{P}^3 = \frac{2^2(\partial_\mu\Phi + \partial_\mu\bar{\Phi})}{4\text{Im } \Phi} + 2gW_\mu\mathcal{P}^3. \tag{4.17}$$

and A_μ is the U(1) connection of the special Kähler manifold.

The main difference with the supersymmetry transformations obtained in [18] on the string configuration is the presence of the axion-dilaton field S coming from the special geometry and parametrizing the manifold $\frac{\text{SU}(1,1)}{\text{U}(1)}$. The gaugini λ_i^S and the U(1) connection A of the axion-dilaton scalar manifold do not distinguish between the two supersymmetry transformations:

- In the supersymmetry transformations of the gravitini fields, the U(1) connection A appears with the same charge for both transformations whereas the matter connection A^B (coming from the SU(2) of the hypermultiplet) comes with opposite charge for the supersymmetries.
- The axion-dilaton field S enters in the same way in the supersymmetric transformations of the gaugini λ_i^S in contrast to the way Φ appears in the supersymmetric transformation of the hyperini.

This difference of behaviour will be more clear in the next section where we analyze the different BPS projectors obtained from the BPS equations.

4.4 Profile of the string

The BPS equations are obtained by setting to zero the supersymmetry transformations (4.16).

In appendix A, we show that a half-BPS solution for a cosmic string solution with magnetic flux requires that ϵ^1 and ϵ^2 should have different chirality on the cosmic string world sheet:

$$\text{BPS projector: } \gamma^{12}\epsilon^1 = \mp i\epsilon^1, \quad \gamma^{12}\epsilon^2 = \pm i\epsilon^2. \tag{4.18}$$

The integrability condition is

$$R_{\mu\nu 12} \pm F_{\mu\nu}^B = 0. \tag{4.19}$$

Using the previous projector the BPS equations read

$$\begin{aligned}
 \left(\partial_\mu \mp \frac{i}{2} \omega_{\mu|12} + \frac{i}{2} A_\mu^B \right) \epsilon^1 &= 0, \\
 \left(\partial_\mu \pm \frac{i}{2} \omega_{\mu|12} - \frac{i}{2} A_\mu^B \right) \epsilon^2 &= 0, \\
 \mp F_{12} + D &= 0, \\
 (\nabla_1 \pm i \nabla_2) \Phi &= 0,
 \end{aligned} \tag{4.20}$$

where $D = 2 \frac{g}{\text{Im} S} \mathcal{P}^3 = -2ge^{-\rho} \mathcal{P}^3$. In appendix A, it is also shown that the BPS equation for the axion-dilaton field implies that it has to be a constant.

The BPS equations (4.20) are the same as those obtained in [18] modulo the factor of $e^{-\rho}$ in the definition of the D -term.

Since the coupling of the Higgs field to the gauge field is non standard (2.31), it is difficult to see what would be the field configuration that corresponds to a cosmic string. To simplify the analysis we define, as in [18], the following field:

$$u = \frac{i - \Phi}{i + \Phi}. \tag{4.21}$$

Under a gauge transformation the field u transforms as:

$$\delta u = 2giu, \tag{4.22}$$

which corresponds to a change of phase. Thus the winding of the phase of u is the one inducing the magnetic flux of the string.

In order to solve the BPS equations we will use the following time independent ansatz:

$$u = f(r)e^{im\theta} \quad W_\theta = W_\theta(r). \tag{4.23}$$

It represents a straight cosmic string of winding m along the z -axis.

Where we have used cylindrical coordinates (t, z, r, θ) .

We take the space-time metric to be of the form:

$$ds^2 = -dt^2 + dz^2 + dr^2 + C^2(r)d\theta^2. \tag{4.24}$$

The BPS equations for the profile of the string are:

$$f'(r) = \pm \frac{f(r)}{C(r)} (m - 4gW_\theta(r)), \quad W_\theta'(r) = \pm C(r)D(r), \quad C'(r) = 1 \mp \tilde{A}_\theta^B(r), \tag{4.25}$$

with

$$\tilde{A}_\theta^B = 2m \frac{f^2}{1-f^2} + e^\rho W_\theta D \quad \text{and} \quad D = -2ge^{-\rho} \left(2 \frac{1+f^2}{1-f^2} - \xi \right). \tag{4.26}$$

From the BPS equations we find the asymptotic behavior of the profile functions. It is similar to the cases of [9, 18].

In the case $r \rightarrow 0$ we have:

$$f(r) \sim r^{\pm n}, \quad C \sim r, \quad W_\theta(r) \sim \pm g(\xi - 2)r^2. \tag{4.27}$$

In the opposite limit, $r \rightarrow \infty$:

$$f(r) \implies \sqrt{\frac{\xi - 2}{\xi + 2}}, \quad W_\theta(r) \implies \frac{m}{4g}, \quad (4.28)$$

and the metric is given by

$$ds^2 = -dt^2 + dz^2 + dr^2 + r^2 \left[1 \mp \frac{1}{2}m(\xi - 2) \right]^2 d\theta^2. \quad (4.29)$$

The upper or lower sign apply for positive or negative winding number m , respectively.

At $r \implies \infty$, the string creates a locally-flat conical metric with a deficit angle proportional to $\xi - 2$. The energy of the string per unit length can be computed as in [9, 18]. One finds that the only non-vanishing contribution comes from the Gibbons-Hawking surface term [58]:

$$\mu_{\text{string}} = - \int d\theta C' \Big|_{r=\infty} + \int d\theta C' \Big|_{r=0} = \pm \pi m(\xi - 2) > 0. \quad (4.30)$$

Note also that the full $N = 2$ supersymmetry is restored asymptotically.

4.5 The fate of the axion-dilaton field

The constant value of the axion-dilaton field is not fixed by the BPS equations nor by the scalar potential. The mass per unit length of the string is also independent of the value of the axion-dilaton field (4.30). The dilaton fixes the overall length scale of the configuration in the following sense. There are two natural lengths in the solution given by the inverse of the masses of the Higgs and the gauge field, and they are both functions of the dilaton field:

$$m_W^2 \propto -\frac{1}{\text{Im } S}, \quad m_\Phi^2 \propto -\frac{1}{\text{Im } S}, \quad (4.31)$$

so that the corresponding length scales are

$$l_W^2 \propto -\text{Im } S, \quad l_\Phi^2 \propto -\text{Im } S. \quad (4.32)$$

Suppose we have a solution to the BPS equations given by the profile functions $f(r)$, $W_\theta(r)$, $C(r)$ and ρ . Then is easy to check that the functions $f(\lambda r)$, $W_\theta(\lambda r)$, $C(\lambda r)/\lambda$ and $\rho - 2 \log(\lambda)$ also satisfy the BPS equations for any real $\lambda > 0$. From here is obvious that the value of the dilaton determines the length scales in the transverse direction to the string, in particular the core radius.

This situation looks similar to the case of semilocal strings [63], were there is also a one parameter family of solutions with equal energy and different core radii. In that case finite energy perturbations can excite the zero mode connecting solutions within the same family, leading to the spread of the magnetic flux and eventually to the disappearance of the strings. This is not going to occur in our model. In order to go from one solution to a different one, the dilaton has to change its value everywhere in the plane transverse to the string. The kinetic energy needed in order to excite the value of the dilaton globally diverges, and this implies that, once the system has chosen a given value for the dilaton, finite energy perturbations cannot drive the system to a solution with a different value of S . The radius of the string will remain unchanged.

5. Discussion

As a generalization of the work done in [18], we have enlarged the family of $\mathcal{N} = 2$ supergravity actions which allow the embedding of $\mathcal{N} = 1$ supergravity actions containing a D -term potential and a constant FI term. We extend the result of [18] to a class of special geometries more familiar in compactifications of string theory. We are using here a “very special Kähler geometry” characterized by a cubic prepotential, instead of the minimal special geometry used in [18]. To be specific we take the special manifold to be:

$$ST[2, n] \equiv \frac{SU(1, 1)}{U(1)} \times \frac{SO(2, n)}{SO(2) \times SO(n)},$$

in the Calabi-Vesentini basis (3.2), which is related to the cubic prepotential by a symplectic rotation [45, 49].

This choice of special geometry has two important consequences. An axion-dilaton field, $S = a - ie^\rho$, is present in the reduced $\mathcal{N} = 1$ theory after truncation from $\mathcal{N} = 2$. Moreover, it is possible to define a gauging for which the scalar potential is bounded from below. However, it has a runaway dependence on the dilaton:

$$\mathcal{V} \propto e^{-\rho}.$$

As an application, we have shown how to construct a half-BPS cosmic string solution from a $\mathcal{N} = 2$ supergravity action in $D = 4$. Following [18] we have used a string ansatz compatible with a consistent truncation from $\mathcal{N} = 2$ to $\mathcal{N} = 1$. In order to obtain the scalar potential we have gauged the same isometry used in [18]. We have found that the BPS equations imply that the axion-dilaton has to be simultaneously holomorphic and anti-holomorphic, which can only be satisfied if it is a constant:

$$S = \text{Constant}, \quad \text{Im}S < 0.$$

Despite the runaway behavior of the potential, we have proved that all the string solutions have the same energy per unit length, regardless of the value of the dilaton, and it is given by the Gibbons-Hawking surface term [57, 35, 18]. The value of the dilaton fixes the masses of the Higgs and the gauge field and, hence, also the radius of the string. We have argued that the system can not evolve between two solutions with different values of the dilaton, since this would require an infinite amount of energy. Thus, once the strings are formed their radii remain fixed.

Observations of the timing of milisecond pulsars give the constraint $\mu_{\text{string}} \lesssim 10^{-7}$ [26]. However this constraint depends on the specific model used to calculate it, what leads to a considerable uncertainty. This implies for our model that the FI term has to satisfy:

$$0 < 2\pi m(\xi - 2) \lesssim 10^{-7},$$

where the lower bound is coming from the study of Minkowski vacua in section 5.1.

An important issue which remains to be discussed is the stability of these strings within the full $\mathcal{N} = 2$ theory. It is far from clear that these strings are stable against perturbations of the truncated fields. Finding such an instability would be an interesting result, since it would provide an example of an unstable BPS solution.

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A. Integrability condition and BPS projector

The integrability conditions of the gravitini BPS equations are:¹⁷

$$(R_{\mu\nu 12}\gamma^{12} + iF_{\mu\nu}^{\mathcal{K}} + iF_{\mu\nu}^B)\epsilon^1 = 0, \quad (R_{\mu\nu 12}\gamma^{12} + iF_{\mu\nu}^{\mathcal{K}} - iF_{\mu\nu}^B)\epsilon^2 = 0, \quad (\text{A.1})$$

To compute the projector we use the following result:¹⁸

$$\left. \begin{array}{l} (a + \gamma^{12}b)\epsilon = 0 \\ \text{or} \\ (a\gamma^1 + b\gamma^2)\epsilon = 0 \end{array} \right\} \xrightarrow{\epsilon \neq 0, a \neq 0} a \mp ib = 0, \quad \gamma^{12}\epsilon = \mp i\epsilon. \quad (\text{A.2})$$

Using the previous method we obtain

$$\mp R_{\mu\nu 12} + (F_{\mu\nu}^{\mathcal{K}} + F_{\mu\nu}^B) = 0, \quad \gamma^{12}\epsilon^1 = \mp i\epsilon^1, \quad (\text{A.3a})$$

$$\mp R_{\mu\nu 12} + (F_{\mu\nu}^{\mathcal{K}} - F_{\mu\nu}^B) = 0, \quad \gamma^{12}\epsilon^2 = \mp i\epsilon^2,$$

$$\mp F_{12} + D = 0, \quad \gamma^{12}\epsilon^1 = \mp i\epsilon^1, \quad (\text{A.3b})$$

$$\mp F_{12} - D = 0, \quad \gamma^{12}\epsilon^2 = \mp i\epsilon^2,$$

$$(\partial_1 \mp i\partial_2)S = 0, \quad \gamma^{12}\epsilon^1 = \mp i\epsilon^1, \quad \gamma^{12}\epsilon^2 = \mp i\epsilon^2, \quad (\text{A.3c})$$

$$*(\nabla_1 \mp i\nabla_2)\Phi = 0, \quad \gamma^{12}\epsilon_1 = \mp i\epsilon_1, \quad (\text{A.3d})$$

$$(\nabla_1 \mp i\nabla_2)\bar{\Phi} = 0, \quad \gamma^{12}\epsilon_2 = \mp i\epsilon_2,$$

¹⁷The integrability conditions of the gravitini are obtained as usual by taking the commutator of two supersymmetries and using the relation

$$[\nabla_\mu, \nabla_\nu] = R_{\mu\nu}.$$

We shall name $F^{\mathcal{K}}$ the curvature of the special geometry U(1) connection A and F^B the curvature of the connection A^B .

¹⁸Given two complex variables a and b and a spinor ϵ , we have

$$(a\gamma^1 \mathbf{1} + b\gamma^2)\epsilon = 0 \implies (a\gamma^1 + b\gamma^2)^2\epsilon = 0 \implies (a^2 + b^2)\epsilon = 0 \xrightarrow{\epsilon \neq 0} a \mp ib = 0.$$

Thus:

$$(a\gamma^1 + b\gamma^2)\epsilon = 0 \xrightarrow{a \neq 0} (\gamma^1 \mp i\gamma^2)\epsilon = 0,$$

where $D = 2\frac{g}{\text{Im}S}\mathcal{P}^3 = -2ge^{-\rho}\mathcal{P}^3$.

Equations (A.3b) are the integrability conditions for the gravitini BPS equations. They imply that $F^{\mathcal{K}}$ (resp. F^B) vanishes if ϵ^1 and ϵ^2 have opposite (resp. the same) chirality on the string world sheet:

$$\begin{aligned} \gamma^{12}\epsilon^1 = \mp i\epsilon^1 \text{ and } \gamma^{12}\epsilon^2 = \mp i\epsilon^2 &\implies F^B = 0, \\ \gamma^{12}\epsilon^1 = \mp i\epsilon^1 \text{ and } \gamma^{12}\epsilon^2 = \pm i\epsilon^2 &\implies F^{\mathcal{K}} = 0, \end{aligned} \tag{A.4}$$

Equations A.3 are compatible in a non-trivial situation (that is for $F_{12} \neq 0$ and $D \neq 0$) if and only if ϵ^1 and ϵ^2 have different chirality on the string world sheet as:

$$\begin{aligned} \gamma^{12}\epsilon^1 = \mp i\epsilon^1 \text{ and } \gamma^{12}\epsilon^2 = \mp i\epsilon^2 &\implies F_{12} = D = 0, \\ \gamma^{12}\epsilon^1 = \mp i\epsilon^1 \text{ and } \gamma^{12}\epsilon^2 = \pm i\epsilon^2 &\implies \mp F_{12} - D = 0. \end{aligned} \tag{A.5}$$

Equations (A.3c) imply that S is a constant if ϵ^1 and ϵ^2 have different chirality whereas S is holomorphic or anti-holomorphic on the plane perpendicular to the axis of the string when ϵ^1 and ϵ^2 have the same chirality on the world sheet:

$$\begin{aligned} \gamma^{12}\epsilon^1 = \mp i\epsilon^1 \text{ and } \gamma^{12}\epsilon^2 = \mp i\epsilon^2 &\implies (\partial_1 \mp \partial_2)S = 0, \\ \gamma^{12}\epsilon^1 = \mp i\epsilon^1 \text{ and } \gamma^{12}\epsilon^2 = \pm i\epsilon^2 &\implies S = \text{Constant}, \end{aligned} \tag{A.6}$$

Finally the hyperini BPS equations are related by complex conjugation. It follows that if ϵ_1 and ϵ_2 have different chirality, Φ is covariantly holomorphic on the plane perpendicular to the string axis. If they have the same chirality it is at the same time holomorphic and anti-holomorphic and therefore constant:

$$\begin{aligned} \gamma^{12}\epsilon^1 = \mp i\epsilon^1 \text{ and } \gamma^{12}\epsilon^2 = \pm i\epsilon^2 &\implies (\nabla_1 \mp i\nabla_2)\Phi = 0, \\ \gamma^{12}\epsilon^1 = \mp i\epsilon^1 \text{ and } \gamma^{12}\epsilon^2 = \mp i\epsilon^2 &\implies \Phi = \text{Constant}. \end{aligned} \tag{A.7}$$

As we are interested in a cosmic string solution involving a magnetic flux ($F \neq 0$) and a non-trivial scalar field Φ ($\neq \text{Constant}$) we shall take the projector which ensures that ϵ^1 and ϵ^2 have different chirality on the string world sheet:

$$\text{BPS projector: } \gamma^{12}\epsilon^1 = \mp i\epsilon^1, \quad \gamma^{12}\epsilon^2 = \pm i\epsilon^2. \tag{A.8}$$

It follows that the axion-dilaton is a constant on the string configuration and that the U(1) connection (A) of the special geometry and its curvature $F^{\mathcal{K}}$ both vanish identically:

$$A = F^{\mathcal{K}} = 0. \tag{A.9}$$

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